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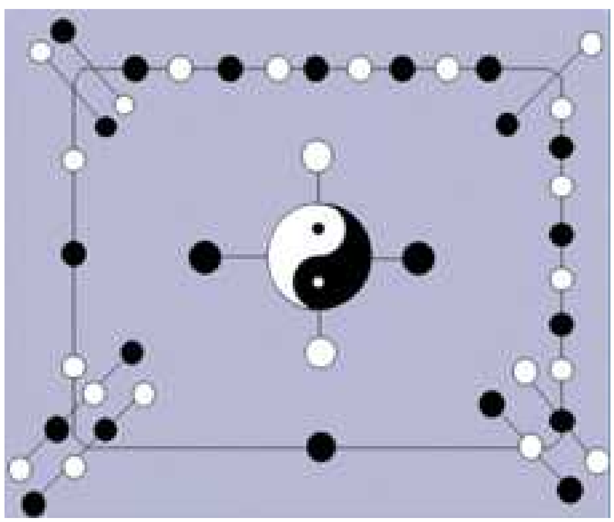
MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



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Aims and Scope: The **Mathematical Combinatorics (International Book Series)** (*ISBN 978-1-59973-040-0*) is a fully refereed international book series and published in USA quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

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Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Applications of Smarandache multi-spaces to theoretical physics; Applications of Combinatorics to mathematics and theoretical physics;

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Combinatorial Speculation and Combinatorial Conjecture for Mathematics

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Abstract Extended: This survey was widely spread after reported at a combinatorial conference of China in 2006. As a powerful tool for dealing with relations among objectives, combinatorics mushroomed in the past century, particularly in catering to the need of computer science and children games. However, an even more important work for mathematician is to apply it to other mathematics and other sciences besides just to find combinatorial behavior for objectives. *How can it contributes more to the entirely mathematical science, not just in various games, but in metric mathematics? What is a right mathematical theory for the original face of our world?* I presented a well-known proverb, i.e., the six blind men and an elephant in *the 3th Northwest Conference on Number Theory and Smarandache's Notion of China* and answered the second question to be Smarandache multi-spaces in logic. Prior to that explaining, I have brought a heartening conjecture for advancing mathematics in 2005, i.e., *mathematical science can be reconstructed from or made by combinatorialization* after a long time speculation, also a bringing about Smarandache multi-space for mathematics. This conjecture is not just like an open problem, but more like a deeply thought for advancing the modern mathematics. The main trend of modern sciences is overlap and hybrid. Whence the mathematics of 21st century should be consistency with the science development in the 21st century, i.e., the *mathematical combinatorics* resulting in the combinatorial conjecture for mathematics. For introducing more readers known this heartening mathematical notion for sciences, there would be no simple stopping point if I began to incorporate the more recent development, for example, the *combinatorially differential geometry*, so it being published here in its original form to survey these thinking and ideas for mathematics and cosmological physics, such as those of multi-spaces, map geometries and combinatorial structures of cosmoses. Some open problems are also included for the advance of 21st mathematics by a combinatorial speculation. More recent progresses can be found in papers and books nearly published, for example, in [20]-[23] for details.

Key words: combinatorial speculation, combinatorial conjecture for mathematics, Smarandache multi-space, M-theory, combinatorial cosmos.

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[2]Reported at *the 2nd Conference on Combinatorics and Graph Theory of China*, Aug. 16-19, 2006, Tianjing, P.R.China

§1. The role of classical combinatorics in mathematics

Modern science has so advanced that to find a universal genus in the society of sciences is nearly impossible. Thereby a scientist can only give his or her contribution in one or several fields. The same thing also happens for researchers in combinatorics. Generally, combinatorics deals with twofold:

Question 1.1. *to determine or find structures or properties of configurations, such as those structure results appeared in graph theory, combinatorial maps and design theory,..., etc..*

Question 1.2. *to enumerate configurations, such as those appeared in the enumeration of graphs, labeled graphs, rooted maps, unrooted maps and combinatorial designs,...,etc..*

Consider the contribution of a question to science. We can separate mathematical questions into three ranks:

Rank 1 *they contribute to all sciences.*

Rank 2 *they contribute to all or several branches of mathematics.*

Rank 3 *they contribute only to one branch of mathematics, for instance, just to the graph theory or combinatorial theory.*

Classical combinatorics is just a *rank* 3 *mathematics* by this view. This conclusion is despair for researchers in combinatorics, also for me 5 years ago. *Whether can combinatorics be applied to other mathematics or other sciences? Whether can it contributes to human's lives, not just in games?*

Although become a universal genus in science is nearly impossible, *our world is a combinatorial world*. A combinatorician should stand on all mathematics and all sciences, not just on classical combinatorics and with a real combinatorial notion, i.e., *combining different fields into a unifying field* ([29]-[32]), such as combine different or even anti-branches in mathematics or science into a unifying science for its freedom of research ([28]). This notion requires us answering three questions for solving a combinatorial problem before. *What is this problem working for? What is its objective? What is its contribution to science or human's society?* After these works be well done, modern combinatorics can applied to all sciences and all sciences are combinatorialization.

§2. The metrical combinatorics and mathematics combinatorialization

There is a prerequisite for the application of combinatorics to other mathematics and other sciences, i.e, to introduce various metrics into combinatorics, ignored by the classical combinatorics since they are the fundamental of scientific realization for our world. This speculation was firstly appeared in the beginning of Chapter 5 of my book [16]:

$\cdots$ *our world is full of measures. For applying combinatorics to other branch of mathematics, a good idea is pullback measures on combinatorial objects again, ignored by the classical combinatorics and reconstructed or make combinatorial generalization for the classical*

mathematics, such as those of algebra, differential geometry, Riemann geometry, Smarandache geometries, $\cdots$ and the mechanics, theoretical physics, $\cdots$.

The combinatorial conjecture for mathematics, abbreviated to *CCM* is stated in the following.

Conjecture 2.1(CCM Conjecture) *Mathematical science can be reconstructed from or made by combinatorialization.*

Remark 2.1 We need some further clarifications for this conjecture.

(1) This conjecture assumes that one can select finite combinatorial rulers and axioms to reconstruct or make generalization for classical mathematics.

(2) Classical mathematics is a particular case in the combinatorialization of mathematics, i.e., the later is a combinatorial generalization of the former.

(3) We can make one combinatorialization of different branches in mathematics and find new theorems after then.

Therefore, a branch in mathematics can not be ended if it has not been combinatorialization and all mathematics can not be ended if its combinatorialization has not completed. There is an assumption in one's realization of our world, i.e., *science can be made by mathematicalization*, which enables us get a similar combinatorial conjecture for the science.

Conjecture 2.2(CCS Conjecture) *Science can be reconstructed from or made by combinatorialization.*

A typical example for the combinatorialization of classical mathematics is the *combinatorial map theory*, i.e., a combinatorial theory for surfaces([14]-[15]). Combinatorially, a surface is topological equivalent to a polygon with even number of edges by identifying each pairs of edges along a given direction on it. If label each pair of edges by a letter $e, e \in \mathcal{E}$, a surface S is also identifying to a cyclic permutation such that each edge $e, e \in \mathcal{E}$ just appears two times in S, one is e and another is e^{-1}. Let $a, b, c, \cdots$ denote the letters in $\mathcal{E}$ and $A, B, C, \cdots$ the sections of successive letters in a linear order on a surface S (or a string of letters on S). Then, a surface can be represented as follows:

$$S = (\cdots, A, a, B, a^{-1}, C, \cdots),$$

where, $a \in \mathcal{E}$,A, B, C denote a string of letters. Define three elementary transformations as follows:

(O_1) $\quad (A, a, a^{-1}, B) \Leftrightarrow (A, B);$

(O_2) $\quad (i) \quad (A, a, b, B, b^{-1}, a^{-1}) \Leftrightarrow (A, c, B, c^{-1});$

$\quad (ii) \quad (A, a, b, B, a, b) \Leftrightarrow (A, c, B, c);$

(O_3) $\quad (i) \quad (A, a, B, C, a^{-1}, D) \Leftrightarrow (B, a, A, D, a^{-1}, C);$

$\quad (ii) \quad (A, a, B, C, a, D) \Leftrightarrow (B, a, A, C^{-1}, a, D^{-1}).$

If a surface S can be obtained from S_0 by these elementary transformations O_1-O_3, we say that S is elementary equivalent with S_0, denoted by $S \sim_{El} S_0$. Then we can get the classification theorem of compact surface as follows([29]):

Any compact surface is homeomorphic to one of the following standard surfaces:

(P_0) the sphere: aa^{-1};

(P_n) the connected sum of $n, n \geq 1$ tori:

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_nb_na_n^{-1}b_n^{-1};$$

(Q_n) the connected sum of $n, n \geq 1$ projective planes:

$$a_1a_1a_2a_2\cdots a_na_n.$$

A *map* M is a connected topological graph cellularly embedded in a surface S. In 1973, Tutte suggested an algebraic representation for an embedding graph on a locally orientable surface ([16]):

A *combinatorial map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a basic permutation $\mathcal{P}$, i.e, for any $x \in \mathcal{X}_{\alpha,\beta}$, no integer k exists such that $\mathcal{P}^k x = \alpha x$, acting on $\mathcal{X}_{\alpha,\beta}$, the disjoint union of *quadricells* Kx of $x \in X$ (the base set), where $K = \{1, \alpha, \beta, \alpha\beta\}$ is the *Klein group* satisfying the following two conditions:

(i) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(ii) *the group* $\Psi_J =< \alpha, \beta, \mathcal{P} >$ *is transitive on* $\mathcal{X}_{\alpha,\beta}$.

For a given map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, it can be shown that $M^* = (\mathcal{X}_{\beta,\alpha}, \mathcal{P}\alpha\beta)$ is also a map, call it the *dual* of the map M. The vertices of M are defined as the pairs of conjugate orbits of $\mathcal{P}$ action on $\mathcal{X}_{\alpha,\beta}$ by the condition (i) and edges the orbits of K on $\mathcal{X}_{\alpha,\beta}$, for example, for $\forall x \in \mathcal{X}_{\alpha,\beta}$, $\{x, \alpha x, \beta x, \alpha\beta x\}$ is an edge of the map M. Define the faces of M to be the vertices in the dual map M^*. Then the Euler characteristic $\chi(M)$ of the map M is

$$\chi(M) = \nu(M) - \varepsilon(M) + \phi(M)$$

where,$\nu(M), \varepsilon(M), \phi(M)$ are the number of vertices, edges and faces of the map M, respectively. For each vertex of a map M, its valency is defined to be the length of the orbits of $\mathcal{P}$ action on a quadricell incident with u.

For example, the graph K_4 on the tours with one face length 4 and another 8 shown in Fig.2.1

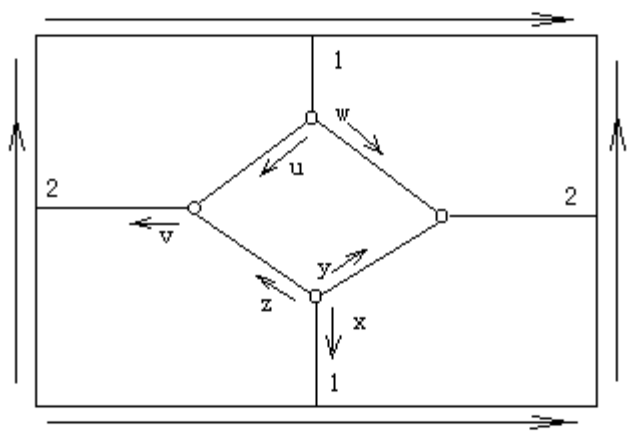


Fig.2.1

can be algebraically represented by $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with $\mathcal{X}_{\alpha,\beta} = \{x, y, z, u, v, w, \alpha x, \alpha y, \alpha z, \alpha u, \alpha v, \alpha w, \beta x, \beta y, \beta z, \beta u, \beta v, \beta w, \alpha\beta x, \alpha\beta y, \alpha\beta z, \alpha\beta u, \alpha\beta v, \alpha\beta w\}$ and

$$\begin{aligned}\mathcal{P} &= (x, y, z)(\alpha\beta x, u, w)(\alpha\beta z, \alpha\beta u, v)(\alpha\beta y, \alpha\beta v, \alpha\beta w)\\ &\times (\alpha x, \alpha z, \alpha y)(\beta x, \alpha w, \alpha u)(\beta z, \alpha v, \beta u)(\beta y, \beta w, \beta v)\end{aligned}$$

with 4 vertices, 6 edges and 2 faces on an orientable surface of genus 1.

By the view of combinatorial maps, these standard surfaces P_0, P_n, Q_n for $n \geq 1$ is nothing but the bouquet B_n on a locally orientable surface with just one face. Therefore, combinatorial maps are the combinatorialization of surfaces.

Many open problems are motivated by the *CCM Conjecture*. For example, a *Gauss mapping* among surfaces is defined as follows.

Let $\mathcal{S} \subset R^3$ be a surface with an orientation $\mathbf{N}$. The mapping $\mathbf{N} : \mathcal{S} \to R^3$ takes its value in the unit sphere

$$S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$$

along the orientation $\mathbf{N}$. The map $\mathbf{N} : \mathcal{S} \to S^2$, thus defined, is called the Gauss mapping.

We know that for a point $P \in \mathcal{S}$ such that the Gaussian curvature $K(P) \neq 0$ and V a connected neighborhood of P with K does not change sign,

$$K(P) = \lim_{A \to 0} \frac{N(A)}{A},$$

where A is the area of a region $B \subset V$ and $N(A)$ is the area of the image of B by the Gauss mapping $N : \mathcal{S} \to S^2$([2],[4]). Now the questions are

(i) *what is its combinatorial meaning of the Gauss mapping? How to realizes it by combinatorial maps?*

(ii) *how can we define various curvatures for maps and rebuilt these results in the classical differential geometry?*

Let $\mathcal{S}$ be a compact orientable surface. Then the *Gauss-Bonnet theorem* asserts that

$$\int\int_{\mathcal{S}} K d\sigma = 2\pi\chi(\mathcal{S}),$$

where K is the Gaussian curvature of $\mathcal{S}$.

By the *CCM Conjecture*, the following questions should be considered.

(*i*) *How can we define various metrics for combinatorial maps, such as those of length, distance, angle, area, curvature,* $\cdots$ *?*

(*ii*) *Can we rebuilt the Gauss-Bonnet theorem by maps for dimensional* 2 *or higher dimensional compact manifolds without boundary?*

One can see references [15] and [16] for more open problems for the classical mathematics motivated by this *CCM Conjecture*, also raise new open problems for his or her research works.

§3. The contribution of combinatorial speculation to mathematics

3.1. The combinatorialization of algebra

By the view of combinatorics, algebra can be seen as a combinatorial mathematics itself. The combinatorial speculation can generalize it by the means of combinatorialization. For this objective, a Smarandache multi-algebraic system is combinatorially defined in the following definition.

Definition 3.1([17],[18]) *For any integers $n, n \geq 1$ and $i, 1 \leq i \leq n$, let A_i be a set with an operation set $O(A_i)$ such that $(A_i, O(A_i))$ is a complete algebraic system. Then the union*

$$\bigcup_{i=1}^{n}(A_i, O(A_i))$$

is called an n multi-algebra system.

An example of multi-algebra systems is constructed by a finite additive group. Now let n be an integer, $Z_1 = (\{0, 1, 2, \cdots, n-1\}, +)$ an additive group (modn) and $P = (0, 1, 2, \cdots, n-1)$ a permutation. For any integer $i, 0 \leq i \leq n-1$, define

$$Z_{i+1} = P^i(Z_1)$$

satisfying that if $k + l = m$ in Z_1, then $P^i(k) +_i P^i(l) = P^i(m)$ in Z_{i+1}, where $+_i$ denotes the binary operation $+_i : (P^i(k), P^i(l)) \to P^i(m)$. Then we know that

$$\bigcup_{i=1}^{n} Z_i$$

is an n multi-algebra system .

The conception of multi-algebra systems can be extensively used for generalizing conceptions and results for these existent algebraic structures, such as those of groups, rings, bodies, fields and vector spaces, $\cdots$, etc.. Some of them are explained in the following.

Definition 3.2 *Let $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ be a closed multi-algebra system with a binary operation set $O(\widetilde{G}) = \{\times_i, 1 \le i \le n\}$. If for any integer $i, 1 \le i \le n$, $(G_i; \times_i)$ is a group and for $\forall x, y, z \in \widetilde{G}$ and any two binary operations $\times$ and $\circ$, $\times \ne \circ$, there is one operation, for example the operation $\times$ satisfying the distribution law to the operation $\circ$ provided their operation results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a multi-group.

For a multi-group $(\widetilde{G}, O(G))$, $\widetilde{G_1} \subset \widetilde{G}$ and $O(\widetilde{G_1}) \subset O(\widetilde{G})$, call $(\widetilde{G_1}, O(\widetilde{G_1}))$ a *sub-multi-group* of $(\widetilde{G}, O(G))$ if $\widetilde{G_1}$ is also a multi-group under the operations in $O(\widetilde{G_1})$, denoted by $\widetilde{G_1} \preceq \widetilde{G}$. For two sets A and B, if $A \bigcap B = \emptyset$, we denote the union $A \bigcup B$ by $A \bigoplus B$. Then we get a generalization of the *Lagrange theorem* of finite group.

Theorem 3.1([18]) *For any sub-multi-group $\widetilde{H}$ of a finite multi-group $\widetilde{G}$, there is a representation set T, $T \subset \widetilde{G}$, such that*

$$\widetilde{G} = \bigoplus_{x \in T} x\widetilde{H}.$$

For a sub-multi-group $\widetilde{H}$ of $\widetilde{G}$, $\times \in O(\widetilde{H})$ and $\forall g \in \widetilde{G}(\times)$, if for $\forall h \in \widetilde{H}$,

$$g \times h \times g^{-1} \in \widetilde{H},$$

then call $\widetilde{H}$ a *normal sub-multi-group* of $\widetilde{G}$. An order of operations in $O(\widetilde{G})$ is said an *oriented operation sequence*, denoted by $\overrightarrow{O}(\widetilde{G})$. We get a generalization of the *Jordan-Hölder theorem* for finite multi-groups.

Theorem 3.2([18]) *For a finite multi-group $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ and an oriented operation sequence $\overrightarrow{O}(\widetilde{G})$, the length of maximal series of normal sub-multi-groups is a constant, only dependent on $\widetilde{G}$ itself.*

In Definition 2.2, choose $n = 2, G_1 = G_2 = \widetilde{G}$. Then $\widetilde{G}$ is a body. If $(G_1; \times_1)$ and $(G_2; \times_2)$ both are commutative groups, then $\widetilde{G}$ is a field. For multi-algebra systems with two or more operations on one set, we introduce the conception of multi-rings and multi-vector spaces in the following.

Definition 3.3 *Let $\widetilde{R} = \bigcup_{i=1}^{m} R_i$ be a closed multi-algebra system with double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i), 1 \le i \le m\}$. If for any integers $i, j,\ i \ne j, 1 \le i, j \le m$, $(R_i; +_i, \times_i)$ is a ring and for $\forall x, y, z \in \widetilde{R}$,*

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

provided all their operation results exist, then $\widetilde{R}$ *is called a multi-ring. If for any integer* $1 \leq i \leq m$, $(R; +_i, \times_i)$ *is a filed, then* $\widetilde{R}$ *is called a multi-filed.*

Definition 3.4 *Let* $\widetilde{V} = \bigcup_{i=1}^{k} V_i$ *be a closed multi-algebra system with binary operation set* $O(\widetilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \leq i \leq m\}$ *and* $\widetilde{F} = \bigcup_{i=1}^{k} F_i$ *a multi-filed with double binary operation set* $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \leq i \leq k\}$. *If for any integers* i, j, $1 \leq i, j \leq k$ *and* $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \widetilde{V}$, $k_1, k_2 \in \widetilde{F}$,

(*i*) $(V_i; \dot{+}_i, \cdot_i)$ *is a vector space on* F_i *with vector additive* $\dot{+}_i$ *and scalar multiplication* $\cdot_i$*;*

(*ii*) $(\mathbf{a} \dot{+}_i \mathbf{b}) \dot{+}_j \mathbf{c} = \mathbf{a} \dot{+}_i (\mathbf{b} \dot{+}_j \mathbf{c})$*;*

(*iii*) $(k_1 +_i k_2) \cdot_j \mathbf{a} = k_1 +_i (k_2 \cdot_j \mathbf{a})$;

provided all those operation results exist, then $\widetilde{V}$ *is called a multi-vector space on the multi-filed* $\widetilde{F}$ *with a binary operation set* $O(\widetilde{V})$, *denoted by* $(\widetilde{V}; \widetilde{F})$.

Similar to multi-groups, we can also obtain results for multi-rings and multi-vector spaces to generalize classical results in rings or linear spaces. Certainly, results can be also found in the references [17] and [18].

3.2. The combinatorialization of geometries

First, we generalize classical metric spaces by the combinatorial speculation.

Definition 3.5 *A multi-metric space is a union* $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ *such that each* M_i *is a space with metric* ρ_i *for* $\forall i, 1 \leq i \leq m$.

We generalized two well-known results in metric spaces.

Theorem 3.3([19]) *Let* $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ *be a completed multi-metric space. For an* ϵ*-disk sequence* $\{B(\epsilon_n, x_n)\}$, *where* $\epsilon_n > 0$ *for* $n = 1, 2, 3, \cdots$, *the following conditions hold:*

(*i*) $B(\epsilon_1, x_1) \supset B(\epsilon_2, x_2) \supset B(\epsilon_3, x_3) \supset \cdots \supset B(\epsilon_n, x_n) \supset \cdots$;

(*ii*) $\lim_{n \to +\infty} \epsilon_n = 0$.

Then $\bigcap_{n=1}^{+\infty} B(\epsilon_n, x_n)$ *only has one point.*

Theorem 3.4([19]) *Let* $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ *be a completed multi-metric space and* T *a contraction on* $\widetilde{M}$. *Then*

$$1 \leq^{\#} \Phi(T) \leq m.$$

Particularly, let $m = 1$. We get the *Banach fixed-point theorem* again.

Corollary 3.1(Banach) *Let M be a metric space and T a contraction on M. Then T has just one fixed point.*

Smarandache geometries were proposed by Smarandache in [29] which are generalization of classical geometries, i.e., these *Euclid, Lobachevshy-Bolyai-Gauss* and *Riemann geometries* may be united altogether in a same space, by some Smarandache geometries under the combinatorial speculation. These geometries can be either partially Euclidean and partially Non-Euclidean, or Non-Euclidean. In general, Smarandache geometries are defined in the next.

Definition 3.6 *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

For example, let us consider an euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual euclidean points on $\mathbf{R}^2$ and s-lines as any euclidean line that passes through one and only one of points A, B and C. Then this geometry is a Smarandache geometry because two axioms are Smarandachely denied comparing with an Euclid geometry:

(i) The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel* and *no parallel*. Let L be an s-line passing through C and is parallel in the euclidean sense to AB. Notice that through any s-point not lying on AB there is one s-line parallel to L and through any other s-point lying on AB there is no s-lines parallel to L such as those shown in Fig.3.1(a).

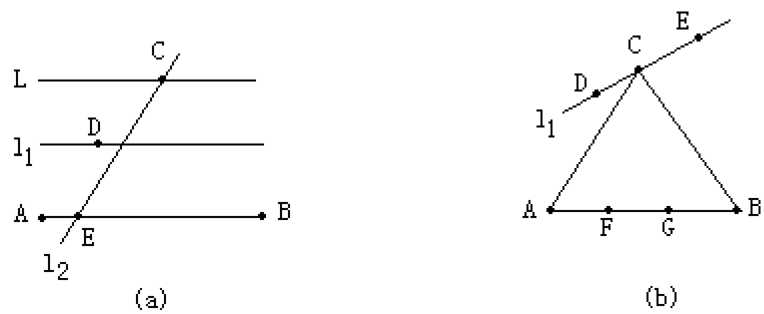


Fig.3.1

(ii) The axiom that through any two distinct points there exists one line passing through them is now replaced by; *one s-line* and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.3.1(b).

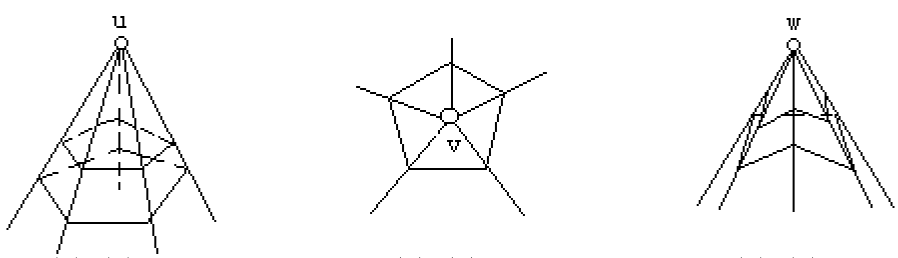
A *Smarandache n-manifold* is an n-dimensional manifold that supports a Smarandache geometry. Now there are many approaches to construct Smarandache manifolds for $n = 2$. A

general way is by the so called *map geometries* without or with boundary underlying orientable or non-orientable maps proposed in references [14] and [15] firstly.

Definition 3.7 *For a combinatorial map M with each vertex valency≥ 3, endow with a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$, to each vertex $u, u \in V(M)$. Call (M, μ) a map geometry without boundary, $\mu(u)$ an angle factor of the vertex u and orientablle or non-orientable if M is orientable or not.*

Definition 3.8 *For a map geometry (M, μ) without boundary and faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M)-1$, if $S(M) \backslash \{f_1, f_2, \cdots, f_l\}$ is connected, then call $(M, \mu)^{-l} = (S(M) \backslash \{f_1, f_2, \cdots, f_l\}, \mu)$ a map geometry with boundary $f_1, f_2, \cdots, f_l$, where $S(M)$ denotes the locally orientable surface underlying map M.*

The realization for vertices $u, v, w \in V(M)$ in a space $\mathbf{R}^3$ is shown in Fig.3.2, where $\rho_M(u)\mu(u) < 2\pi$ for the vertex u, $\rho_M(v)\mu(v) = 2\pi$ for the vertex v and $\rho_M(w)\mu(w) > 2\pi$ for the vertex w, are called to be elliptic, euclidean or hyperbolic, respectively.



$\rho_M(u)\mu(u) < 2\pi$ $\rho_M(u)\mu(u) = 2\pi$ $\rho_M(u)\mu(u) > 2\pi$

Fig.3.2

On an Euclid plane $\mathbf{R}^2$, a straight line passing through an elliptic or a hyperbolic point is shown in Fig.3.3.

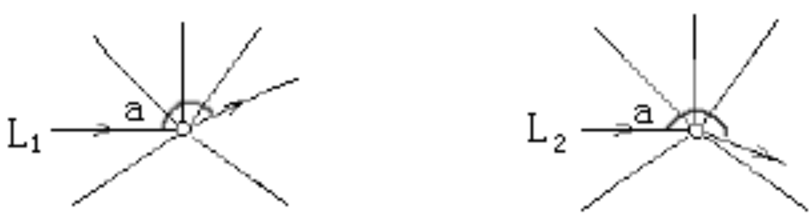


Fig.3.3

Theorem 3.5([17]) *There are Smarandache geometries, including paradoxist geometries, non-geometries and anti-geometries in map geometries without or with boundary.*

Generally, we can ever generalize the ideas in Definitions 3.7 and 3.8 to a metric space and find new geometries.

Definition 3.9 *Let U and W be two metric spaces with metric ρ, $W \subseteq U$. For $\forall u \in U$, if there is a continuous mapping $\omega : u \to \omega(u)$, where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any number $\epsilon > 0$, there exists a number $\delta > 0$ and a point $v \in W$, $\rho(u - v) < \delta$ such that $\rho(\omega(u) - \omega(v)) < \epsilon$, then U is called a metric pseudo-space if $U = W$ or a bounded metric pseudo-space if there is a number $N > 0$ such that $\forall w \in W$, $\rho(w) \leq N$, denoted by (U, ω) or (U^-, ω), respectively.*

For the case $n = 1$, we can also explain $\omega(u)$ being an angle function with $0 < \omega(u) \leq 4\pi$ as in the case of map geometries without or with boundary, i.e.,

$$\omega(u) = \begin{cases} \omega(u)(mod 4\pi), & \text{if u} \in \text{W}, \\ 2\pi, & \text{if u} \in \text{U} \setminus \text{W} \qquad (*) \end{cases}$$

and get some interesting metric pseudo-space geometries. For example, let $U = W =$ Euclid plane $= \sum$, then we obtained some interesting results for pseudo-plane geometries $(\sum, \omega)$ as shown in the following([17]).

Theorem 3.6 *In a pseudo-plane $(\sum, \omega)$, if there are no euclidean points, then all points of $(\sum, \omega)$ is either elliptic or hyperbolic.*

Theorem 3.7 *There are no saddle points and stable knots in a pseudo-plane plane $(\sum, \omega)$.*

Theorem 3.8 *For two constants ρ_0, θ_0, $\rho_0 > 0$ and $\theta_0 \neq 0$, there is a pseudo-plane $(\sum, \omega)$ with*

$$\omega(\rho, \theta) = 2(\pi - \frac{\rho_0}{\theta_0 \rho}) \text{ or } \omega(\rho, \theta) = 2(\pi + \frac{\rho_0}{\theta_0 \rho})$$

such that

$$\rho = \rho_0$$

is a limiting ring in $(\sum, \omega)$.

Now for an m-manifold M^m and $\forall u \in M^m$, choose $U = W = M^m$ in Definition 3.9 for $n = 1$ and $\omega(u)$ a smooth function. We get a pseudo-manifold geometry (M^m, ω) on M^m. By definitions in the reference [2], a *Minkowski norm* on M^m is a function $F : M^m \to [0, +\infty)$ such that

(i) F is smooth on $M^m \setminus \{0\}$;
(ii) F is 1-homogeneous, i.e., $F(\lambda\overline{u}) = \lambda F(\overline{u})$ for $\overline{u} \in M^m$ and $\lambda > 0$;
(iii) for $\forall y \in M^m \setminus \{0\}$, the symmetric bilinear form $g_y : M^m \times M^m \to R$ with

$$g_y(\overline{u}, \overline{v}) = \frac{1}{2} \frac{\partial^2 F^2(y + s\overline{u} + t\overline{v})}{\partial s \partial t}|_{t=s=0}$$

is positive definite and a *Finsler manifold* is a manifold M^m endowed with a function $F : TM^m \to [0, +\infty)$ such that

(*i*) F is smooth on $TM^m \setminus \{0\} = \bigcup\{T_{\overline{x}}M^m \setminus \{0\} : \overline{x} \in M^m\}$;
(*ii*) $F|_{T_{\overline{x}}M^m} \to [0, +\infty)$ is a Minkowski norm for $\forall \overline{x} \in M^m$.

As a special case, we choose $\omega(\overline{x}) = F(\overline{x})$ for $\overline{x} \in M^m$, then (M^m, ω) is a *Finsler manifold*. Particularly, if $\omega(\overline{x}) = g_{\overline{x}}(y, y) = F^2(x, y)$, then (M^m, ω) is a *Riemann manifold*. Therefore, we get a relation for Smarandache geometries with Finsler or Riemann geometry.

Theorem 3.9 *There is an inclusion for Smarandache, pseudo-manifold, Finsler and Riemann geometries as shown in the following:*

$$\begin{aligned}\{Smarandache\ geometries\} &\supset \{pseudo-manifold\ geometries\}\\ &\supset \{Finsler\ geometry\}\\ &\supset \{Riemann\ geometry\}.\end{aligned}$$

Other purely mathematical results on the combinatorially differential geometry, particularly the combinatorially Riemannian geometry can be found in recently finished papers [20] – [23] of mine.

§4. The contribution of combinatorial speculation to theoretical physics

The progress of theoretical physics in last twenty years of the 20th century enables human beings to probe the mystic cosmos: *where are we came from? where are we going to?*. Today, these problems still confuse eyes of human beings. Accompanying with research in cosmos, new puzzling problems also arose: *Whether are there finite or infinite cosmoses? Are there just one? What is the dimension of the Universe? We do not even know what the right degree of freedom in the Universe is*, as Witten said([3]).

We are used to the idea that our living space has three dimensions: *length, breadth* and *height*, with time providing the fourth dimension of spacetime by Einstein. Applying his principle of general relativity, i.e. *all the laws of physics take the same form in any reference system* and equivalence principle, i.e., *there are no difference for physical effects of the inertial force and the gravitation in a field small enough.*, Einstein got the *equation of gravitational field*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}.$$

where $R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}_{\mu i\nu}$,

$$R^{\alpha}_{\mu i\nu} = \frac{\partial \Gamma^i_{\mu i}}{\partial x^{\nu}} - \frac{\partial \Gamma^i_{\mu\nu}}{\partial x^i} + \Gamma^{\alpha}_{\mu i}\Gamma^i_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^i_{\alpha i},$$

$$\Gamma^g_{mn} = \frac{1}{2}g^{pq}(\frac{\partial g_{mp}}{\partial u^n} + \frac{\partial g_{np}}{\partial u^m} - \frac{\partial g_{mn}}{\partial u^p})$$

and $R = g^{\nu\mu}R_{\nu\mu}$.

Combining the Einstein's equation of gravitational field with the *cosmological principle*, i.e., *there are no difference at different points and different orientations at a point of a cosmos*

on the metric $10^4 l.y.$, *Friedmann* got a standard model of cosmos. The metrics of the standard cosmos are

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]$$

and

$$g_{tt} = 1,\ g_{rr} = -\frac{R^2(t)}{1-Kr^2}, g_{\phi\phi} = -r^2R^2(t)\sin^2\theta.$$

The standard model of cosmos enables the birth of big bang model of the Universe in thirties of the 20th century. The following diagram describes the developing process of the Universe in different periods after the Big Bang.

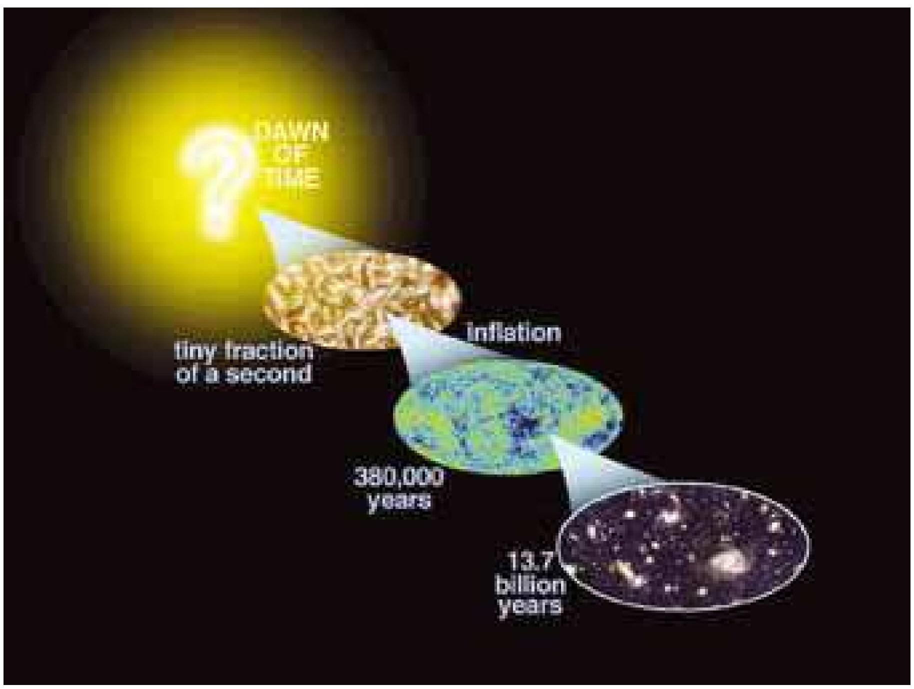


Fig.4.1

4.1**. The M-theory**

The M-theory was established by Witten in 1995 for the unity of those five already known string theories and superstring theories, which postulates that all matter and energy can be reduced to *branes* of energy vibrating in an 11 dimensional space, then in a higher dimensional space solve the Einstein's equation of gravitational field under some physical conditions ([1],[3],[26]-[27]). Here, a *brane* is an object or subspace which can have various spatial dimensions. For any integer $p \geq 0$, a p-*brane* has length in p dimensions. For example, a 0-*brane* is just a point or particle; a 1-*brane* is a string and a 2-*brane* is a surface or membrane, $\cdots$.

We mainly discuss line elements in differential forms in Riemann geometry. By a geometrical view, these p-branes in M-theory can be seen as *volume elements in spaces*. Whence, we

can construct a graph model for p-branes in a space and combinatorially research graphs in spaces.

Definition 4.1 *For each m-brane $\mathbf{B}$ of a space $\mathbf{R}^m$, let $(n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B}))$ be its unit vibrating normal vector along these p directions and $q: \mathbf{R}^m \to \mathbf{R}^4$ a continuous mapping. Now construct a graph phase $(\mathcal{G}, \omega, \Lambda)$ by*

$$V(\mathcal{G}) = \{p - branes\ q(\mathbf{B})\},$$

$$E(\mathcal{G}) = \{(q(\mathbf{B}_1), q(\mathbf{B}_2)) | there\ is\ an\ action\ between\ \mathbf{B}_1\ and\ \mathbf{B}_2\},$$

$$\omega(q(\mathbf{B})) = (n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B})),$$

and

$$\Lambda(q(\mathbf{B}_1), q(\mathbf{B}_2)) = \ forces\ between\ \mathbf{B}_1\ and\ \mathbf{B}_2.$$

Then we get a graph phase $(\mathcal{G}, \omega, \Lambda)$ in $\mathbf{R}^4$. Similarly, if $m = 11$, it is a graph phase for the M-theory.

As an example for applying M-theory to find an accelerating expansion cosmos of 4-dimensional cosmoses from supergravity compactification on hyperbolic spaces is the *Townsend-Wohlfarth type metric* in which the line element is

$$ds^2 = e^{-m\phi(t)}(-S^6dt^2 + S^2dx_3^2) + r_C^2 e^{2\phi(t)} ds_{H_m}^2,$$

where

$$\phi(t) = \frac{1}{m-1}(\ln K(t) - 3\lambda_0 t),$$

$$S^2 = K^{\frac{m}{m-1}} e^{-\frac{m+2}{m-1}\lambda_0 t}$$

and

$$K(t) = \frac{\lambda_0 \zeta r_c}{(m-1)\sin[\lambda_0 \zeta |t + t_1|]}$$

with $\zeta = \sqrt{3 + 6/m}$. This solution is obtainable from space-like brane solution and if the proper time ς is defined by $d\varsigma = S^3(t)dt$, then the conditions for expansion and acceleration are $\frac{dS}{d\varsigma} > 0$ and $\frac{d^2S}{d\varsigma^2} > 0$. For example, the expansion factor is 3.04 if $m = 7$, i.e., a really expanding cosmos.

According to M-theory, the evolution picture of our cosmos started as a perfect 11 dimensional space. However, this 11 dimensional space was unstable. The original 11 dimensional space finally cracked into two pieces, a 4 and a 7 dimensional subspaces. The cosmos made the 7 of the 11 dimensions curled into a tiny ball, allowing the remaining 4 dimensions to inflate at enormous rates, the Universe at the final.

4.2. The combinatorial cosmos

The combinatorial speculation made the following combinatorial cosmos in the reference [17].

Definition 4.2 *A combinatorial cosmos is constructed by a triple* (Ω, Δ, T), *where*

$$\Omega = \bigcup_{i\geq 0} \Omega_i, \quad \Delta = \bigcup_{i\geq 0} O_i$$

and $T = \{t_i; i \geq 0\}$ *are respectively called the cosmos, the operation or the time set with the following conditions hold.*

(1) (Ω, Δ) *is a Smarandache multi-space dependent on* T, *i.e., the cosmos* (Ω_i, O_i) *is dependent on time parameter* t_i *for any integer* $i, i \geq 0$.

(2) *For any integer* $i, i \geq 0$, *there is a sub-cosmos sequence*

$$(S): \ \Omega_i \supset \cdots \supset \Omega_{i1} \supset \Omega_{i0}$$

in the cosmos (Ω_i, O_i) *and for two sub-cosmoses* (Ω_{ij}, O_i) *and* (Ω_{il}, O_i), *if* $\Omega_{ij} \supset \Omega_{il}$, *then there is a homomorphism* $\rho_{\Omega_{ij},\Omega_{il}} : (\Omega_{ij}, O_i) \to (\Omega_{il}, O_i)$ *such that*

(*i*) *for* $\forall(\Omega_{i1}, O_i), (\Omega_{i2}, O_i), (\Omega_{i3}, O_i) \in (S)$, *if* $\Omega_{i1} \supset \Omega_{i2} \supset \Omega_{i3}$, *then*

$$\rho_{\Omega_{i1},\Omega_{i3}} = \rho_{\Omega_{i1},\Omega_{i2}} \circ \rho_{\Omega_{i2},\Omega_{i3}},$$

where $\circ$ *denotes the composition operation on homomorphisms.*

(*ii*) *for* $\forall g, h \in \Omega_i$, *if for any integer* i, $\rho_{\Omega,\Omega_i}(g) = \rho_{\Omega,\Omega_i}(h)$, *then* $g = h$.

(*iii*) *for* $\forall i$, *if there is an* $f_i \in \Omega_i$ *with*

$$\rho_{\Omega_i,\Omega_i \bigcap \Omega_j}(f_i) = \rho_{\Omega_j,\Omega_i \bigcap \Omega_j}(f_j)$$

for integers $i, j, \Omega_i \bigcap \Omega_j \neq \emptyset$, *then there exists an* $f \in \Omega$ *such that* $\rho_{\Omega,\Omega_i}(f) = f_i$ *for any integer* i.

By this definition, there is just one cosmos Ω and the sub-cosmos sequence is

$$\mathbf{R}^4 \supset \mathbf{R}^3 \supset \mathbf{R}^2 \supset \mathbf{R}^1 \supset \mathbf{R}^0 = \{P\} \supset \mathbf{R}_7^- \supset \cdots \supset \mathbf{R}_1^- \supset \mathbf{R}_0^- = \{Q\}.$$

in the string/M-theory. In Fig.4.2, we have shown the idea of the combinatorial cosmos.

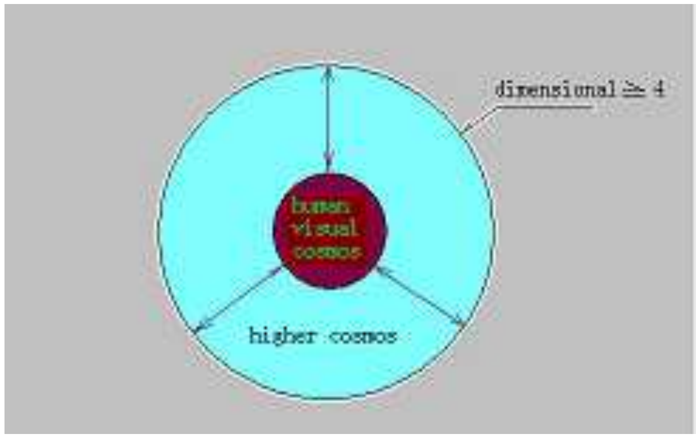


Fig.4.2

For 5 or 6 dimensional spaces, it has been established a dynamical theory by this combinatorial speculation([24]-[25]). In this dynamics, we look for a solution in the Einstein's equation of gravitational field in 6-dimensional spacetime with a metric of the form

$$ds^2 = -n^2(t,y,z)dt^2 + a^2(t,y,z)d\sum_k^2 + b^2(t,y,z)dy^2 + d^2(t,y,z)dz^2$$

where $d\sum_k^2$ represents the 3-dimensional spatial sections metric with $k = -1, 0, 1$ respective corresponding to the hyperbolic, flat and elliptic spaces. For 5-dimensional spacetime, deletes the indefinite z in this metric form. Now consider a 4-brane moving in a 6-dimensional *Schwarzschild-ADS spacetime*, the metric can be written as

$$ds^2 = -h(z)dt^2 + \frac{z^2}{l^2}d\sum_k^2 + h^{-1}(z)dz^2,$$

where

$$d\sum_k^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega_{(2)}^2 + (1-kr^2)dy^2$$

and

$$h(z) = k + \frac{z^2}{l^2} - \frac{M}{z^3}.$$

Then the equation of a 4-dimensional cosmos moving in a 6-spacetime is

$$2\frac{\ddot{R}}{R} + 3(\frac{\dot{R}}{R})^2 = -3\frac{\kappa_{(6)}^4}{64}\rho^2 - \frac{\kappa_{(6)}^4}{8}\rho p - 3\frac{\kappa}{R^2} - \frac{5}{l^2}$$

by applying the *Darmois-Israel conditions* for a moving brane. Similarly, for the case of $a(z) \neq b(z)$, the equations of motion of the brane are

$$\frac{d^2\dot{d}\dot{R}-d\ddot{R}}{\sqrt{1+d^2\dot{R}^2}}-\frac{\sqrt{1+d^2\dot{R}^2}}{n}(d\dot{n}\dot{R}+\frac{\partial_z n}{d}-(d\partial_z n-n\partial_z d)\dot{R}^2)=-\frac{\kappa_{(6)}^4}{8}(3(p+\rho)+\hat{p}),$$

$$\frac{\partial_z a}{ad}\sqrt{1+d^2\dot{R}^2}=-\frac{\kappa_{(6)}^4}{8}(\rho+p-\hat{p}),$$

$$\frac{\partial_z b}{bd}\sqrt{1+d^2\dot{R}^2}=-\frac{\kappa_{(6)}^4}{8}(\rho-3(p-\hat{p})),$$

where the energy-momentum tensor on the brane is

$$\hat{T}_{\mu\nu}=h_{\nu\alpha}T_\mu^\alpha-\frac{1}{4}Th_{\mu\nu}$$

with $T_\mu^\alpha=diag(-\rho,p,p,p,\hat{p})$ and the *Darmois-Israel conditions*

$$[K_{\mu\nu}]=-\kappa_{(6)}^2\hat{T}_{\mu\nu},$$

where $K_{\mu\nu}$ is the extrinsic curvature tensor.

The combinatorial cosmos also presents new questions to combinatorics, such as:

(i) to embed a graph into spaces with dimensional≥ 4;
(ii) to research the phase space of a graph embedded in a space;
(iii) to establish graph dynamics in a space with dimensional≥ 4, $\cdots$, etc..

For example, we have gotten the following result for graphs in spaces in [17].

Theorem 4.1 *A graph G has a nontrivial including multi-embedding on spheres $P_1\supset P_2\supset\cdots\supset P_s$ if and only if there is a block decomposition $G=\biguplus\limits_{i=1}^{s}G_i$ of G such that for any integer $i, 1<i<s$,*

(i) *G_i is planar;*

(ii) *for $\forall v\in V(G_i)$, $N_G(x)\subseteq(\bigcup\limits_{j=i-1}^{i+1}V(G_j))$.*

Further research of the combinatorial cosmos will richen the knowledge of combinatorics and cosmology, also get the combinatorialization for cosmology.

References

[1] I.Antoniadis, Physics with large extra dimensions: String theory under experimental test, *Current Science*, Vol.81, No.12, 25(2001),1609-1613

[2] S.S.Chern and W.H.Chern, *Lectures in Differential Geometry*(in Chinese), Peking University Press, 2001.

[3] M.J.Duff, A layman's guide to M-theory, *arXiv*: hep-th/9805177, v3, 2 July(1998).

[4] B.J.Fei, *Relativity Theory and Non-Euclid Geometries*, Science Publisher Press, Beijing, 2005.

[5] U.Günther and A.Zhuk, Phenomenology of brane-world cosmological models, *arXiv: gr-qc/0410130*.

[6] S.Hawking, *A Brief History of Times*, A Bantam Books/November, 1996.

[7] S.Hawking, *The Universe in Nutshell*, A Bantam Books/November, 2001.

[8] D.Ida, Brane-world cosmology, *arXiv: gr-qc/9912002*.

[9] H.Iseri, *Smarandache Manifolds*, American Research Press, Rehoboth, NM,2002.

[10] H.Iseri, *Partially Paradoxist Smarandache Geometries*, http://www.gallup.unm. edu/~smarandache/Howard-Iseri-paper.htm.

[11] M.Kaku, *Hyperspace: A Scientific Odyssey through Parallel Universe, Time Warps and 10th Dimension*, Oxford Univ. Press.

[12] P.Kanti, R.Madden and K.A.Olive, A 6-D brane world model, *arXiv: hep-th/0104177*.

[13] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *Mathematics Magazine, Aurora, Canada*, Vol.12(2003)

[14] L.F.Mao, *On Automorphisms groups of Maps, Surfaces and Smarandache geometries*, *Sientia Magna*, Vol.1(2005), No.2, 55-73.

[15] L.F.Mao, A new view of combinatorial maps by Smarandache's notion, in *Selected Papers on Mathematical Combinatorics*(I), World Academic Union, 2006, also in *arXiv: math.GM/0506232*.

[16] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[17] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.

[18] L.F.Mao, On algebraic multi-groups, *Sientia Magna*, Vol.2(2006), No.1, 64-70.

[19] L.F.Mao, On multi-metric spaces, *Sientia Magna*, Vol.2(2006), No.1, 87-94.

[20] L.F.Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[21] L.F.Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[22] L.F.Mao, A combinatorially generalized Stokes' theorem on integrations, *International J.Mathematical Combinatorics*, Vol.1, No.1(2007),67-86.

[23] L.F.Mao, Curvature equations on combinatorial manifolds with applications to theoretical physics, *International J.Mathematical Combinatorics* (accepted).

[24] E.Papantonopoulos, Braneworld cosmological models, *arXiv: gr-qc/0410032*.

[25] E.Papantonopoulos, Cosmology in six dimensions, *arXiv: gr-qc/0601011*.

[26] J.A.Peacock, *Cosmological Physics*, Cambridge University Press, 2003.

[27] J.Polchinski, *String Theory*, Vol.1-Vol.2, Cambridge University Press, 2003.

[28] D.Rabounski, Declaration of academic freedom: Scientific Human Rights, *Progress in Physics*, January (2006), Vol.1, 57-60.

[29] F.Smarandache, Mixed non-euclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

[30] F.Smarandache, *A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[31] F.Smarandache, Neutrosophy, a new Branch of Philosophy, *Multi-Valued Logic*, Vol.8, No.3(2002)(special issue on Neutrosophy and Neutrosophic Logic), 297-384.

[32] F.Smarandache, A Unifying Field in Logic: Neutrosophic Field, *Multi-Valued Logic*, Vol.8, No.3(2002)(special issue on Neutrosophy and Neutrosophic Logic), 385-438.

[33] J.Stillwell, *Classical topology and combinatorial group theory*, Springer-Verlag New York Inc., (1980).

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Structures of Cycle Bases with Some Extremal Properties

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Abstract: In this paper, we investigate the structures of cycle bases with extremal properties which are related with *map geometries*, i.e., *Smarandache* 2*-dimensional manifolds*. We first study the long cycle base structures in a cycle space of a graph. Our results show that much information about long cycles is contained in a longest cycle base. (1)Any two longest cycle bases have the same structure, i.e., there is a 1-1 correspondence between any two longest cycle bases such that the corresponding cycles have the same length; (2)Any group of linearly independent longest cycles must be contained in a longest cycle base which implies that any two sets of linearly independent longest cycles with maximum cardinal number is equivalent; (3)If consider the range of embedded graphs, a longest cycle base must contain some long cycles with special properties. As applications, we find explicit formulae for computing longest cycles bases of several class of embedded graphs. As for an embedded graph on non-orientable surfaces, we obtain several interpolation results for one-sided cycles in distinct cycle bases. Similar results for shortest cycle bases may be deduced. For instance, we show that in a strongly embedded graph, there is a cycle base consisting of surface induced non-separating cycles and all of such bases have the same structure provided that their length is of shortest(subject to induced non-separating cycles). These extend Tutte's result [7](which states that in a 3–connected graph the set of induced(graph) non-separating cycles generate the cycle space).

Keywords: Cycle space, longest cycle base, SDR, long cycle.
AMS(2000): 05C30.

§1. Introduction

Here in this paper we consider connected graphs without loops. Concepts and terminologies used without definition may be found in [1]. A spanning subgraph H of G is called an *E-subgraph* iff each vertex has even degree in H. It is well known that the set of E-subgraphs of G forms a linear space $\mathscr{C}(G)$ called the *cycle space* of G. Here, the operation between vectors(i.e., E-subgraphs) is the symmetric difference between edge-sets of E-subgraphs. It is clear that the rank, defined by $\beta(G)$ (the Betti number of G), of $\mathscr{C}(G)$ is $|E(G)| - |V(G)| + 1$ and any set of $\beta(G)$ linearly independence vectors form a base of $\mathscr{C}(G)$. The length $l(\mathcal{B})$ of a cycle base $\mathcal{B}$

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is the sum of length of vectors in it. In particular, the length of an E-subgraph is the sum of length of edge-disjoint cycles in it. Throughout this paper, we only consider the vectors with only one cycle. So, the bases considered are all formed by cycles. By a longest base $\mathcal{B}$ we mean $l(\mathcal{B})$ is the length of a maximum cycle base.

Cycle space theory rooted in early research works of Kirchoff's circuits theory. In theory, Matroid theory is one of motivations of it [10-12], also related with map geometries, i.e., Smarandache 2-dimensional manifolds ([5]-[6]). In particular, cycle bases with minimum length have many applications in structural analysis [2], chemical storage theory [3], as well as fields such bioscience [4]. In history, classical works concentrated on minimum cycle bases(i.e., MCB). On the other direction, results for cycle spaces theory on long cycles are seldom to be seen. What can we say about longest cycle bases? In intuition, a longest cycle base should contain information about long cycles(especially the longest cycles). Here, in this paper we investigate the structure of longest cycle bases. Based on a Hall type theorem for base transformation, we present a condition for a cycle base to be longest.

Theorem A *Let $\mathcal{B}$ be a cycle base(i.e., vectors of $\mathcal{B}$ are all cycles) of G. Then $\mathcal{B}$ is longest if and only if for every cycle C of G:*

$$\forall\, \alpha \in \mathrm{Int}(C) \Longrightarrow |\,\alpha\,| \geq |\,C\,| \tag{1}$$

where $\mathrm{Int}(C)$ *is the set of cycles in $\mathcal{B}$ which span C.*

Note: (1) This condition says that for a longest base $\mathcal{B}$, any cycle can't be generated by shorter cycles of $\mathcal{B}$;

(2) One may see that such Hall type theorem is very useful in studies of cycle bases with particular extremal properties.

The following result shows that any group of linearly independent longest cycles are contained in a longest cycle base. In particular, any longest cycle is contained in a longest cycle base.

Theorem B *Let $C_1, C_2, \ldots, C_s$ be a set of linearly independent longest cycles of graph G. Then there is a longest cycle base $\mathcal{B}$ containing $C_i, 1 \leq i \leq s$.*

If consider the cycles passing through an edge, then after using Theorem A we may see that for every edge e of a graph G, every longest cycle base must contain a cycle which is longest among cycles passing through e.

Corollary 1 *Any longest cycle of a graph must be contained in a longest cycle base.*

Based on Theorem A, we obtain the following unique structure of longest cycle bases.

Theorem C *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of two longest cycle bases of a graph G. Then there is a* 1-1 *correspondence φ between $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for each cycle $\alpha \in \mathcal{B}_1$, $|\,\varphi(\alpha)| = |\,\alpha|$.*

Corollary 2 *A graph G's any two longest cycle bases must contain the same number of k-cycles, for k = 3, 4, …,n.*

Since the condition (1) of Theorem A implies that a cycle can't be generated by shorter

cycles in a longest cycle base, we have the following

Corollary 3 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of two longest cycle bases of a graph G. Then the two subgroups of $\mathcal{B}_1$ and $\mathcal{B}_2$ which contain longest cycles are linearly equivalent.*

Corollary 4 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of two longest cycle bases of a graph G and $A_k, A_k^{'}$ be the sets of k-cycles of $\mathcal{B}_1$ and $\mathcal{B}_2$, resp. Then $\bigcup\limits_{k=p}^{n} A_k$ is equivalent to $\bigcup\limits_{k=p}^{n} A_k^{'}$, for each p = 3, 4, $\ldots$,n.*

As applications of Theorems A-C, we will compute the length of longest cycle bases in several types of graphs. But what surprises us most is that those results are also very useful in computing cycle bases with particular extremal properties. In particular, we have the following

Theorem D *Let G be an embedded graph with $\mathcal{B}_1$ and $\mathcal{B}_2$ to be a pair of its longest(shortest) cycles bases. If $\mathcal{B}_1$ and $\mathcal{B}_2$ contain, resp., s and t distinct one-sided cycles, then there is a longest(shortest) cycle base $\mathcal{B}$ with exactly k distinct one-sided cycles for every integer k between s and t.*

Since our results may be applied to any pair of bases, we have

Theorem D$^{'}$ *Let G be an embedded graph, and $\mathcal{B}_1, \mathcal{B}_2$ be a pair of cycle bases containing, resp., m and n one-sided cycles. Then G has a cycle base containing exactly k distinct one-sided cycles for any natural number k between m and n.*

A cycle C of an embedded graph G in a surface $\sum$ is called *(surface)non-separating* if $\sum - C$ is connected; otherwise, it is *(surface)separating*. If one component of $\sum - C$ is an open disc, then C is *contractible* or *trivial*; if not so, C is called *non-contractible*. It is clear that a non-separating cycle is also non-contractible. Since a non-separating cycle can't be spanned by separating cycles (as we will show later), we have the following result.

Theorem E *A longest cycle base of an embedded graph must contain a longest non-separating cycle; any longest non-separating cycle is also contained in a longest cycle base; furthermore, if a pair of longest cycle bases contains, respectively, m and n longest non-separating cycles, then for every integer $k : m \leq k \leq n$, there is a longest cycle base containing exactly k longest non-separating cycles .*

On the other direction, if we consider the shortest cycle bases, then interesting properties on short cycles will appear. We call a graph G in a surface to be *LEW-embedded* if the length of shortest non-contractible cycle is longer than any facial walk. It is well known that an LEW-embedded graph shares many properties with planar graphs [8]. Here, we will present some more unknown results for cycle bases of LEW-embedded graphs.

Theorem F *Let G be an LEW-embedded graph and $\mathcal{B}_1, \mathcal{B}_2$ be a pair of shortest cycle bases. Then, we have the following results:*
(1) For any separating cycle $C \in \mathcal{B}_i$ and non-separating cycle $C^{'} \in \mathcal{B}_i$, $|C^{'}| > |C|$;
(2) Both $\mathcal{B}_1$ and $\mathcal{B}_2$ contain exactly $\nu(\sum)$ non-separating cycles, where $\nu(\sum)$ is the Euler-genus of the surface $\sum$ in which G is embedded; further more, the subsets of separating cycles of $\mathcal{B}_1$

and $\mathcal{B}_2$ are linearly equivalent;
(3) Both $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same number of shortest non-separating cycles.

If we restrict some condition on an embedded graph, then some unknown results are obtained. For instance, we have the following

Theorem G *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of longest cycle bases of an embedded graph G. If the length of longest non-separating cycle is longer than that of any separating cycle, then both $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same number of longest non-separating cycles.*

A cycle of a graph is *induced* if it has no chord. A famous result in cycle space theory is due to W. Tutte which states that in a simple 3–connected graph, the set of induced cycles each of which can't separate the graph generates the whole cycle space [9]. If we consider the case of embedded graphs, then this cycle set may be smaller. In fact, we have the following

Theorem *Let G be a 2–connected graph embedded in a non-spherical surface such that its facial walks are all cycles. Then there is a cycle base consists of induced non-separating cycles.*

Remark(1) Tutte's definition of a non-separating cycle differs from ours. The former defined a cycle which can't separate the graph, while the latter define a cycle which can't separate the surface in which the graph is embedded. So, Theorem H and Tutte's result are different. From our proof one may see that this base is determined simply by (surface)non-separating cycles. As for the structure of such bases, we may modify the condition of Theorem A and obtain another condition for bases consisting of shortest non-separating cycles.

Remark(2) Theorem H implies the existence of a cycle base $\mathcal{B}$ satisfying

i) All cycles in this cycle base $\mathcal{B}$ are non-separating;
ii) The length of this base $\mathcal{B}$ is shortest subject to i).

We call a base defined above as *shortest non-separating cycle base*.

Theorem I *Let G be a 2–connected graph embedded in a non-spherical surface such that all of its facial walks are cycles. Let $\mathcal{B}$ be a base consisting of non-separating cycles. Then $\mathcal{B}$ is shortest iff for every non-separating cycle C,*

$$\forall\ \alpha \in \mathrm{Int}(C) \Rightarrow |\,C| \geq |\,\alpha|,$$

where $\mathrm{Int}(C)$ is the subset of cycles of $\mathcal{B}$ which span C.

Combining Theorems H and I we obtain the following unique structure result for shortest non-separating cycle bases.

Theorem J *Let G be a 2–connected graph embedded in some non-spherical surface with all its facial walks as cycles. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of shortest non-separating cycle bases. Then there exists a 1–1 correspondence φ between elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for every element $\alpha \in \mathcal{B}_1$, $|\,\alpha| = |\,\varphi(\alpha)|$.*

Remark From our proof of Theorem J, one may see that if the surface in which the graph is embedded is non-orientable, then we may find a cycle base consisting of one–sided cycles and

so, there is a cycle base satisfying

i) All cycles in the base are one–sided cycles;

ii) The length of the base is shortest subject to i);

iii) Any pair of cycle bases satisfying i) and ii) have the same structure, i.e., there is a 1–1 correspondence between them such that the corresponding cycles have the same length.

§2. Proofs of general results

In this section we shall prove Theorems A – C. Firstly, we should set up some preliminaries works. Let $\mathcal{M} = (S_1, S_2, \ldots, S_m)$ be a set of m sets. If each S_i contains an element a_i such that $a_i \neq a_j$ for $i \neq j$, then $(a_1, a_2, \ldots, a_m)$ is called a SDR of $\mathcal{M}$. The following is a famous condition for a system of sets to have a SDR.

Lemma 1(Hall's theorem [7]) *Let $\mathcal{M} = (S_1, S_2, \ldots, S_m)$ be a system of sets. Then $\mathcal{M}$ has a SDR iff for any k subsets of $\mathcal{M}$, their union has at least k elements, $1 \leq k \leq m$.*

The following is an application of Lemma 1.

Lemma 2 *Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}, \mathcal{B}_2 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be a pair of bases of a linearly vector space $\mathcal{V}_m$ over a field $\mathscr{F}$. Then $\mathcal{M} = (S_1, S_2, \ldots, S_m)$ has a SDR, where $S_i = \mathrm{Int}(\alpha_i)$ is the set of vectors of $\mathcal{B}_2$ which spans $\alpha_i, 1 \leq i \leq m$.*

Proof Suppose on the contrary. Then there is an integer number k and k subsets, say $S_1, S_2, \ldots, S_k$ such that

$$\left| \bigcup_{i=1}^{k} S_i \right| < k \tag{2}$$

This shows that $\alpha_1, \alpha_2, \ldots, \alpha_k$ may be generated by less than k elements of $\mathcal{B}_2$, a contradiction as desired. □

Proof of Theorem A Let $\mathcal{B}$ be a longest cycle base of G and C be a cycle of G. Then there is a set $\mathrm{Int}(C)$ of cycles of $\mathcal{B}$ which span C, i.e., $C = \sum\limits_{C_i \in \mathrm{C}} \oplus\, C_i$. If there is a cycle $C_i \in \mathrm{Int}(C)$ with $|\, C_i| < |\, C|$, then $\mathcal{B}_1 = \mathcal{B} - C_i + C$ is another cycle base with length longer than that of $\mathcal{B}_1$, contrary to the definition of $\mathcal{B}$. Thus, (1) holds for every cycle of G. On the other hand, suppose that $\mathcal{B}$ is a cycle base of G satisfying (1) and $\mathcal{B}_1$ is a longest cycle base of G. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\},\ \mathcal{B}_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_m\},\ m = \beta(G)$. Then for each $\gamma_i \in \mathcal{B}_1$, there is a set $\mathrm{Int}(\gamma_i)$ of cycles of $\mathcal{B}$ which span γ_i. By Lemma 2, $(\mathrm{Int}(\gamma_1), \mathrm{Int}(\gamma_2), \ldots, \mathrm{Int}(\gamma_m))$ has a SDR$= (\alpha_1^{'}, \alpha_2^{'}, \ldots, \alpha_m^{'})$ such that $\alpha_i^{'} \in \mathrm{Int}(\gamma_i), 1 \leq i \leq m$. Then by (1), we have that

$$|\, \alpha_i^{'}| \geq |\, \gamma_i|\,, \quad 1 \leq i \leq m$$

which implies that $l(\mathcal{B}) \geq l(\mathcal{B}_1)$ and so, $\mathcal{B}$ is also a longest cycle base of G. □

Proof of Theorem B Let $\mathcal{B}$ be a longest cycle base of G such that $|\, \mathcal{B} \cap \{C_1, C_2, \ldots, C_s\}|$ is as large as possible. If $|\, \mathcal{B} \cap \{C_1, C_2, \ldots, C_s\}| = s$, then $C_i \in \mathcal{B}$ for $1 \leq i \leq s$. $\mathcal{B}$ is the right cycle base. Otherwise, there is an integer $k\,(1 \leq k \leq s)$ such that $C_k \notin \mathcal{B}$. Then $\mathcal{B}$ has a subset $\mathrm{Int}(C_k)$ spanning C_k. It is clear that $\mathrm{Int}(C_k) \nsubseteq \{C_1, C_2, \ldots, C_s\}$. Hence, there is a

cycle $C_j \in \mathrm{Int}(C_k)\backslash\{C_1, C_2, \ldots, C_s\}$. Since Theorem A shows that a cycle can't be generated by shorter cycles in a longest cycle base, we have that $|C_j| = |C_k|$. Thus, $\mathcal{B}_1 = \mathcal{B} - C_j + C_k$ is a longest cycle base containing more cycles in $\{C_1, C_2, \ldots, C_s\}$ than that of $\mathcal{B}$, a contradiction as desired. $\square$

Proof of Theorem C Let $\mathcal{B}_1 = \{C_1, C_2, \ldots, C_m\}$, $\mathcal{B}_2 = \{C_1', C_2', \ldots, C_m'\}$ be a pair of longest cycle bases of G, $m = \beta(G)$. Then for each $C_i' \in \mathcal{B}_2$, there is a subset $\mathrm{Int}(C_i') \subseteq \mathcal{B}_1$ such that C_i' is spanned by vectors of $\mathrm{Int}(C_i')$. By Lemma 2, $\big(\mathrm{Int}(C_1'), \mathrm{Int}(C_2'), \ldots, \mathrm{Int}(C_m')\big)$ has a SDR, say $(C_1, C_2, \ldots, C_m)$ with $C_i \in \mathrm{Int}(C_i')$, $1 \leq i \leq m$. By Theorem A, $|C_i'| \leq |C_i|$, $1 \leq i \leq m$. Let $\varphi : C_i \longmapsto C_i'$. Then φ is a 1-1 correspondence between $\mathcal{B}_1$ and $\mathcal{B}_2$. Since both of them are longest, we have that $|\varphi(C_i)| = |C_i'| = |C_i|$, $1 \leq i \leq m$. This ends the proof of Theorem C. $\square$

§3. Applications to embedded Graphs

In this section, we shall apply the results of § 2 to obtain some important results in graph theory. We first introduce some definition for graph embedding. Let G be a graph which is topologically embedded in a surface S such that each component of $S - G$ is an open disc. Such graph embedding are called *2-cell embedding*. We may also define such embedding in another way as the monograph [8] did. An embedding of a graph is a rotations system $\pi = \{\pi_v | \, v \in V(G)\}$ (each π_v is a cyclic permutation of semi-edges around v) with a signature $\pi : E(G) \longmapsto \{-1, 1\}$. If a cycle C has even-number of negative signatures, it is called a *two-sided cycle*; otherwise, it is called a *one-sided cycle*. If an embedding permits no one-sided cycles, then it is called an *orientable embedding*; otherwise, it is *non-orientable embedding*. It is clear that a one-sided cycle is contained in a Möbius band which bounds a crosscap.

Proof of Theorem D Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $\mathcal{B}_2 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be a pair of longest(shortest) cycle bases of a graph G, $m = \beta(G)$, such that $\mathcal{B}_1$ and $\mathcal{B}_2$ have s and t one-sided cycles, resp. Suppose that $s < t$ and k is an integer : $s \leq k \leq t$. We will show that there exists a longest cycle base $\mathcal{B}$ with exactly k one-sided cycles. We apply induction on the value of $|s - t|$. It is clear that the result holds for smaller value. Now suppose that it holds for values smaller than $|s - t|$. By Lemma 2, $(\mathrm{Int}(\beta_1), \mathrm{Int}(\beta_2), \ldots, \mathrm{Int}(\beta_m))$ has a SDR, say $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m})$ with $\alpha_{i_j} \in \mathrm{Int}(\beta_j)$, where each $\mathrm{Int}(\beta_j)$ is the set of cycles of $\mathcal{B}_1$ which span $\beta_j, 1 \leq j \leq m$. Further, $|\alpha_{i_j}| = |\beta_j|$ by the definition of $\mathcal{B}_1$ and $\mathcal{B}_2, 1 \leq j \leq m$. Since $\mathcal{B}_2$ has more one-sided cycles than that of $\mathcal{B}_1$, there is a one-sided cycle β_j such that $\mathrm{Int}(\beta_j)$ contains a two-sided cycle, say α_j', of $\mathcal{B}_1$. In fact, we may choose $\alpha_{i_j} = \alpha_j'$ by the 1-1 correspondence. Now let $\mathcal{B} = \mathcal{B}_1 - \alpha_{i_j} + \beta_j$. Then $\mathcal{B}$ is another longest cycle base with exactly $s + 1$ one-sided cycles. By induction hypothesis, the result holds. $\square$

Proof of Theorem D$'$ It follows from the proof of Theorem D. $\square$

Before our proving of Theorem E, we have to do some preliminary works. First, we have the following result for surface topology.

Lemma 3 *Let G be an embedded graph and C a non-separating cycle of G. Then C can't be*

generated by a group of separating cycles .

Proof Since every separating cycle is two–sided and a one–sided cycle can't be spanned by two–sided cycles, we may suppose that C is a two-sided non-separating cycle. Recall that C is non-separating iff $G_l(C) = G_r(C)$, where $G_l(C)$ and $G_r(C)$ are, respectively, the left–subgraph and right–subgraph of C (as defined in [8]). Suppose that C may be spanned by a set of separating cycles. Then C may also by spanned by a set of facial walks : $\partial f_1, \partial f_2, \ldots, \partial f_s$, i.e.,

$$C = \partial f_1 \oplus \partial f_2 \oplus \ldots \oplus \partial f_s\,, \qquad \mathrm{Int}(C) = \{\, \partial f_1, \partial f_2, \ldots, \partial f_s \,\},$$

This implies that for every edge e of C , e is covered(contained) in exactly one facial walk of $\mathrm{Int}(C) = \{\, \partial f_1, \partial f_2, \ldots, \partial f_s \,\}$ and every edge in $\{\, \partial f_1, \partial f_2, \ldots, \partial f_s \,\} \backslash E(C)$ is contained in exactly two walks in $\{\, \partial f_1, \partial f_2, \ldots, \partial f_s \,\}$.

Let $x \in V(C)$ and e be an edge of C containing x . Then the local rotation of edges incident to x is $\Pi_x = (\, e, e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_q)$, where e_{p+1} is another edge of C having a common vertex with e. Each pair of consecutive edges forms a corner $\angle e_i\, x e_{i+1}$ containing x. It is clear that each corner is contained in a region bounded by some facial walk in $\mathrm{Int}(C)$. If the corner $\angle exe_1$ is contained in a region bounded by a facial walk, then each corner $\angle e_i\, x e_{i+1}\, (1 \le i \le p)$ is also contained in some facial walk. In particular, e_{p+1} is also contained in a facial walk. Thus, if a facial walk of $\mathrm{Int}(C)$ is on the right–hand side of C and shares an edge with C, then all corner together with its edges on the right–side of C are contained in facial walks of $\mathrm{Int}(C)$. Since each edge of C is contained in exactly one facial walk of $\mathrm{Int}(C)$, we see that no facial walk of $\mathrm{Int}(C)$ may contain an edge of C which is in $G_l(C)$. Notice that C is non-separating and thus there is an path P starting from an edge of $G_r(C)$ containing a vertex of C and ending at another edge in $G_l(C)$ which contains a vertex of C. This implies that G^*, the dual graph of G, contains a path P^* connecting a pair of facial walks which are on the distinct side of C. We may choose P^* such that it has no edge corresponding to an edge of C. It is easy to see that the vertices of P^* correspond to a set of facial walks of $\mathrm{Int}(C)$ which form a facial walk chain. Hence, the two end–facial walks corresponding to the two end–vertices of P must be in $\mathrm{Int}(C)$. This is impossible since $\mathrm{Int}(C)$ has no such pair of facial walks (containing edges in C) on distinct side of C. This ends the proof of Lemma 3. □

Proof of Theorem E Let $\mathcal{B}$ be a longest cycle base and C a longest non-separating cycle. If $C \notin \mathcal{B}$, then C is spanned by a set $\mathrm{Int}(C)$ of cycles of $\mathcal{B}$. By Lemma 3, $\mathrm{Int}(C)$ contains a non-separating cycle $C^{'}$ which is no shorter than that of C (by (1) of Theorem A), so $|\,C| = |\,C^{'}|$ and $C^{'}$ is also a longest non-separating cycle. This proves the first part of Theorem E. Now let

$$\begin{aligned}\mathcal{B}_1 &= \{\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_{\beta(G)}\},\\ \mathcal{B}_2 &= \{\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_{\beta(G)}\},\end{aligned}$$

be a pair of longest cycle bases with exactly m and n non-separating cycles. Let $\alpha_i (1 \le i \le m)$ and $\gamma_j (1 \le j \le n)$ be non-separating cycles of $\mathcal{B}_1$ and $\mathcal{B}_1$, respectively. Then for each $\gamma_i \in \mathcal{B}_2$, there is a set $\mathrm{Int}(\gamma_i)$ of cycles of $\mathcal{B}_1$ spanning γ_i. By the proving procedure of Theorem A, the

system of sets

$$\big(\,\mathrm{Int}(\gamma_1), \mathrm{Int}(\gamma_2), \ldots, \mathrm{Int}(\gamma_n), \ldots, \mathrm{Int}(\gamma_{\beta(G)})\,\big)$$

has a SDR $(\,\alpha_1^{'}, \alpha_2^{'}, \ldots, \alpha_n^{'}, \ldots, \alpha_{\beta(G)}^{'}\,)$ and further $\alpha_i^{'} \in \mathrm{Int}(\gamma_i)$ such that $|\,\alpha_i^{'}| = |\,\gamma_i|, 1 \le i \le \beta(G)$. It is clear that there is an integer, say $k(1 \le k \le n)$, such that $\alpha_k^{'}$ is separating since $m < n$ implies that $\mathcal{B}_2$ has more longest non-separating cycle than that of $\mathcal{B}_1$. Now consider the set $\mathcal{B}_3 = \mathcal{B}_2 - \gamma_k + \alpha_k^{'}$ is a longest cycle base containing exactly $n-1$ longest non-separating cycles. Repeating this procedure, we may find a longest cycle base with exactly l longest non-separating cycles for each $l : m \le l \le n$. This ends the proof of Theorem E. $\square$

Proof of Theorem F Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_{\beta(G)}\}$ be a MCB (minimum cycle base) of an LEW-embedded graph G, where $\alpha_i(1 \le i \le m)$ and $\alpha_j(m < j \le \beta(G))$ are, respectively, non-separating cycle and separating cycle. Suppose that there are φ facial walks: $\partial f_1, \partial f_2, \ldots, \partial f_\varphi$. It is clear that $\alpha_{m+1}, \alpha_{m+2}, \ldots,\ \alpha_{\beta(G)}$ may be linearly expressed by $\{\partial f_1, \partial f_2, \ldots, \partial f_{\varphi-1}\}$. Let $\partial f_i(1 \le i \le \varphi - 1)$ be a facial walk. Then ∂f_i is spanned by a subset $\mathrm{Int}(\partial f_i)$ of $\mathcal{B}_1$. Since $\mathcal{B}_1$ is shortest, every cycle of $\mathrm{Int}(\partial f_i)$ must be contractible by Theorem A. Thus, $\{\partial f_1, \partial f_2, \ldots, \partial f_{\varphi-1}\}$ is linearly equivalent to $\{\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_{\beta(G)}\}$, i.e., $\beta(G) - m = \varphi - 1$(which says that $\mathcal{B}_1$ has exactly $\nu(\sum)$ non-separating cycles, where $\nu(\sum)$ is the Euler-genus of the host surface $\sum$ on which G is embedded). This ends the proof of (2).

Let α_i and α_j be, respectively, non-separating cycle and separating cycle of $\mathcal{B}_1$ such that $|\,\alpha_i| \le |\,\alpha_j|$. Then α_j is spanned by a set $\mathrm{Int}(\alpha_j)$ of facial walks. It is clear that there is a facial walk, say α_k, of $\mathrm{Int}(\alpha_j)$ which can't be generated by vectors in $\mathcal{B}_1 \backslash \{\alpha_j\}$. It is easy to see that $|\,\partial f_k| < |\,\alpha_j|$(since otherwise, $|\,\alpha_i| \le |\,\partial f_k|$ will contrary to the definition of LEW-embedded graph). Hence, $\mathcal{B}_1 - \alpha_j + \partial f_k$ will be a shorter cycle base, contrary to the definition of $\mathcal{B}_1$. So, we have $|\,\alpha_i| > |\,\alpha_j|$ which ends the proof of (1).

Let

$$\begin{aligned} \mathcal{B}_1 &= \{\alpha_1, \alpha_2, \ldots, \alpha_s, \alpha_{s+1}, \ldots, \alpha_{\beta(G)}\}, \\ \mathcal{B}_2 &= \{\gamma_1, \gamma_2, \ldots, \gamma_t, \gamma_{t+1}, \ldots, \gamma_{\beta(G)}\}, \end{aligned}$$

be a pair of MCBs such that $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ and $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ are, respectively, the set of longest non-separating cycles of $\mathcal{B}_1$ and $\mathcal{B}_2$. Suppose that $s \le t$. Then for each $\gamma_i(1 \le i \le \beta(G))$, there is a subset $\mathrm{Int}(\gamma_i)$ of $\mathcal{B}_1$ which span γ_i. By the proving procedure of Theorem A, the system of sets: $(\mathrm{Int}(\gamma_1), \mathrm{Int}(\gamma_2), \ldots, \mathrm{Int}(\gamma_{\beta(G)}))$ has a SDR, say $(\alpha_1^{'}, \alpha_2^{'}, \ldots, \alpha_{\beta(G)}^{'})$ such that $\alpha_i^{'} \in \mathrm{Int}(\gamma_i)$ and $|\,\alpha_i^{'}| = |\,\gamma_i|\,, 1 \le i \le \beta(G)$. By (1) we see that each $\alpha_i^{'}(1 \le i \le t)$ is non-separating which implies that $\alpha_1^{'}, \alpha_2^{'}, \ldots, \alpha_t^{'}$ is a collection of longest non-separating cycles of G in $\mathcal{B}_1$. Thus, $t \le s$. This ends the proof of (3). $\square$

Proof of Theorem G It follows from the proving procedure of Theorem E. $\square$

Proof of Theorem H Notice that any cycle base consists of two parts: the first part is determined by non-separating cycles while the second part is composed of separating cycles. So, what we have to do is to show that any facial cycle may be generated by non-separating cycles. Our proof depends on two steps.

Step 1 Let x be a vertex of G. Then there is a non-separating cycle passing through x.

Let $C^{'}$ be a non-separating cycle of G which avoids x. Then by Menger's theorem, there are two inner disjoint paths P_1 and P_2 connecting x and $C^{'}$. Let $P_1 \cap C^{'} = \{u\}$, $P_2 \cap C^{'} = \{v\}$. Suppose further that $u\overrightarrow{C^{'}}v$ and $v\overrightarrow{C^{'}}u$ are two segments of $C^{'}$, where $\overrightarrow{C}$ is an orientation of C. Then there are three inner disjoint paths connecting u and v:

$$Q_1 = u\overrightarrow{C}v, \quad Q_2 = v\overrightarrow{C}u, \quad Q_3 = P_1 \cup P_2\,.$$

Since $C^{'} = Q_1 \cup Q_2$ is non-separating, at least one of cycles $Q_2 \cup Q_3$ and $Q_1 \cup Q_3$ is non-separating by Lemma 3.

Step 2 Let ∂f be any facial cycle. Then there exist two non-separating cycles C_1 and C_2 which span ∂f.

In fact, we add a new vertex x into the inner region of $\partial f(i.e., int(\partial f))$ and join new edges to each vertex of ∂f. Then the resulting graph also satisfies the condition of Theorem H. By Step 1, there is a non-separating C passing through x. Let u and v be two vertices of $C \cap \partial f$. Then $u\overrightarrow{C}v$ together with two segments of ∂f connecting u and v forms a pair of non-separating cycles. □

Proof of Theorem I and J It follows from the proving procedure of Theorem A and C. □

§4. Examples

Next, we will compute the lengths of longest cycle bases in some types of graphs.

Example 1 Let G be a " Möbius ladder graph " embedded in the projective plane as shown in Fig.1.

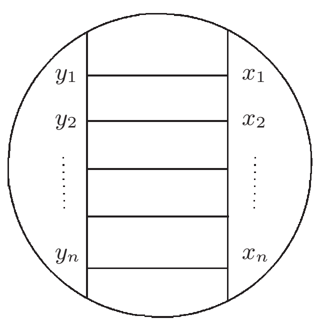

Fig. 1

It is clear that G is non-planar and 3-regular. There are n quadrangles defined as

$$C_4^{(i)} = \begin{cases} (x_i, x_{i+1}, y_{i+1}, y_i)\,, & 1 \le i \le n-1 \\ \\ (x_n, y_1, x_1, y_n)\,, & i = n \end{cases}$$

and n Hamiltonian cycles as

$$H_i = \begin{cases} H - \{(x_i, x_{i+1}), (y_i, y_{i+1})\} + \{(x_i, y_i), (x_{i+1}, y_{i+1})\}, & 1 \leq i \leq n-1 \\ \\ (x_1, x_2, \ldots, x_n, y_n, y_{n-1}, \ldots, y_2, y_1), & i = n \end{cases}$$

where H is the Hamiltonian cycle $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$. It is easy to see that $C_4^{(i)} \oplus H_i$ is the Hamiltonian cycle H.

Case 1 $n \equiv 0\,(mod\ 2)$.

Claim 1 *$\{H_1, H_2, \ldots, H_n\}$ is a linearly independent set.*

If not so, one may see that

$$H_1 \oplus H_2 \oplus \cdots \oplus H_n = 0$$

This implies that

$$\left(H_1 \oplus C_4^1\right) \oplus \left(H_2 \oplus C_4^2\right) \oplus \cdots \oplus \left(H_n \oplus C_4^n\right) = C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^n$$

i.e.,

$$n\,H = H = 0\,,$$

a contradiction.

Let C be a (2n-1)-cycle which is non-contractible. Since $n \equiv 0\,(mod\ 2)$, we have

Claim 2 *C can't be generated by $\{H_1, H_2, \ldots, H_n\}$.*

This follows from the fact that C is a one-sided cycle which can't be spanned by two-sided cycles. Then $\mathcal{B} = \{C, H_1, H_2, \ldots, H_n\}$ is a longest cycle base. Otherwise, G would have a longest cycle base which consists of $n+1$ Hamiltonian cycles, and so G is bipartite. This is a contradiction with the fact that G has an odd cycle $(x_1, x_2, \ldots, x_n, y_n)$.

Case 2 $n \equiv 1\,(mod\ 2)$

Claim 3 *$\{H_1, H_2, \ldots, H_{n-1}\}$ is a set of linearly independent cycles.*

This time, we consider the contractible Hamiltonian cycle H. Then $\{H_1, H_2, \ldots, H_{n-1}, H\}$ is also a set of linearly independent cycles. If not so, H would be the sum of $H_1, H_2, \ldots, H_{n-1}$, i.e.,

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_{n-1}\,,$$

that is,

$$\begin{aligned} H \oplus C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^{n-1} &= \left(H_1 \oplus C_4^1\right) \oplus \left(H_2 \oplus C_4^2\right) \oplus \left(H_{n-1} \oplus C_4^{n-1}\right) \\ &= (n-1)\,H = \ 0\,, \end{aligned}$$

Now, we have that

$$H = C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^{n-1}\,.$$

This is impossible(since $C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^{n-1} \oplus C_4^n = H$).

Let $H^{'}$ be a non-contractible Hamiltonian cycle. Then by Claim 2, $\mathcal{B} = \{H_1, H_2, \ldots, H_{n-1}, H, H^{'}\}$ is a Hamiltonian base of G.

Example 2 Let us consider the longest cycle base of $\mathcal{K}_n$, the complete graph with n vertices. It is easy to see that $\beta(\mathcal{K}_n) = \frac{1}{2}(n-1)(n-2) = C_{n-1}^{\,2}$, which suggests us to give a combinatorial explanation of $\beta(\mathcal{K}_n)$. Suppose $V(G) = \{x_1, x_2, \ldots, x_n\}$. Then $\mathcal{K}_n - x_n = \mathcal{K}_{n-1}$, i.e., the complete graph with $n-1$ vertices $x_1, x_2, \ldots, x_{n-1}$. Let us consider a (n-1)-cycle $\overrightarrow{C}_{n-1} = (x_1, x_2, \ldots, x_{n-1})$ and $x_i,\ x_j \in V(C_{n-1})(i < j)$. Then $H_{i,j} = x_{i-1}\overleftarrow{C}C_{n-1}x_j\, x_i\overrightarrow{C}_{n-1}x_{j-1}$ is a Hamiltonian path of $\mathcal{K}_{n-1}$. Now we find $\beta(\mathcal{K}_n)$ Hamiltonian cycles defined as $C_n(i, j) = (x_n\, x_{i-1}\overleftarrow{C}C_{n-1}x_j\, x_i\overrightarrow{C}_{n-1}x_{j-1})$ in formal.

Claim 4 *If* $|\,i-j| \geq 2$, *then the set* $\{C_n(i, j)|1 \leq i < j \leq n-1\}$ *is linearly independent set.*

This follows frow the fact that $(x_i,\, x_j) \in E(C_n(i,\, j))$ is an edge which can't be deleted by the definition of symmetric difference.

Case 1 $n \equiv 1\,(mod\ 2)$

Now the n -cycles $C_n(i,\, i+1) = (x_n, x_{i+1}, x_{i+2}, \ldots, x_{n-1}, x_1, x_2, \ldots, x_i), (1 \leq i \leq n-1)$ is linearly independent cycles. Otherwise, we have that

$$C_n(1,\, 2) \oplus C_n(2,\, 3) \oplus \ldots \oplus C_n(n-1,\, 1) = 0$$

which implies $\cap C_{n-1} = 0$, a contradiction! Based on this and Claim 4, $\{C_n(i,\, j)|1 \leq i < j \leq n-1\}$ is a set of linearly independent Hamiltonian cycles.

Case 2 $n \equiv 0\,(mod\ 2)$

Although $\{C_n(1,\, 2), C_n(2,\, 3), \ldots, C_n(n,\, 1)\}$ is linearly dependent set of Hamilton cycles, $\{C_n(1,\, 2), C_n(2,\, 3), \ldots, C_n(n-1,\, n)\}$ is a set of linearly independent cycles. Since $\mathcal{K}_n$ can't have a Hamiltonian base, it's longest cycle base is $\{C_n(i,\, j)|1 \leq i < j \leq n\} \backslash \{C_n(n,\, 1)\}$ together with a (n-1)-cycle (1, 2, ...,n - 1).

Example 3 Let G be an outer planar triangular graph embedded in the sphere with its triangular faces $f_1, f_2, \ldots, f_{\varphi-1}$. Then it has exactly one Hamiltonian cycle ∂f_φ, here we use ∂f to denote the boundary of a face f. By Euler's formula, $\varphi - 1 = \beta(G)$, where φ is the number of faces. Let us define a set of cycles as following

$$C_n = \partial f_\varphi,$$

$$C_{n-1} = \partial f_1 \oplus \partial f_2 \oplus \cdots \oplus \partial f_{\varphi-2}, \quad C_{n-1}^{'} = \partial f_{\varphi-1} \oplus \partial f_{\varphi-2} \oplus \cdots \oplus \partial f_2$$

$$C_{n-2} = \partial f_1 \oplus \partial f_2 \oplus \cdots \oplus \partial f_{\varphi-3}, \quad C_{n-2}^{'} = \partial f_{\varphi-1} \oplus \partial f_{\varphi-2} \oplus \cdots \oplus \partial f_3$$

$$C_{n-k} = \partial f_1 \oplus \partial f_2 \oplus \cdots \oplus \partial f_{\varphi-k-1}$$

$$C_{n-k}^{'} = \partial f_{\varphi-1} \oplus \partial f_{\varphi-2} \oplus \cdots \oplus \partial f_{k+1}, \quad 1 \leq k \leq \varphi - 2.$$

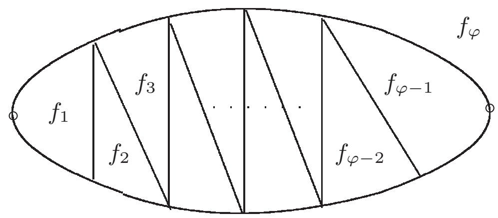

Fig. 2

$$\mathcal{B}=\begin{cases}\{C_n,C_{n-1},C_{n-2},\ldots,C_{\frac{n+3}{2}}\}\cup\{C'_{n-1},C'_{n-2},\ldots,C'_{\frac{n+3}{2}}\}, & \varphi\equiv 0\,(mod\ 2)\\ \\ \{C_n,C_{n-1},C_{n-2},\ldots,C_{\frac{n+4}{2}}\}\cup\{C'_{n-1},C'_{n-2},\ldots,C'_{\frac{n+2}{2}}\}, & \varphi\equiv 1\,(mod\ 2)\end{cases}$$

Thus $\mathcal{B}$ satisfies the condition of Theorem A. Hence, $\mathcal{B}$ is a longest cycle base, and the length of longest cycle base is

$$l(B)=\begin{cases}n+2(n-1)+2(n-2)+\cdots+2\left(\dfrac{n+3}{2}\right), & \varphi\equiv 0\,(mod\ 2)\\ \\ n+2(n-1)+2(n-2)+\cdots+2\left(\dfrac{n+4}{2}\right)+\dfrac{n+2}{2}, & \varphi\equiv 1\,(mod\ 2)\end{cases}$$

Example 4 Again we consider the “ Möbius ladder graph ” in Fig.1. It is clear that the edge–width(i.e., $ew(G)$) is $n+1$ and there are $n+1$ shortest non-separating cycles:

$$C_i=\begin{cases}(y_1,y_2,\ldots,y_i,x_i,x_{i+1},\ldots,x_n), & 1\le i\le n\\ \\ (y_1,y_2,\ldots,y_n,x_1), & i=n+1\end{cases}$$

Notice that $\beta(G)=n+1$ and $\{C_1,C_2,\ldots,C_{n+1}\}$ may generate every facial cycle and every non-contractible cycle of G. Thus, $\mathcal{B}=\{C_1,C_2,\ldots,C_n\}$ is a shortest non-separating cycle base with length $l(\mathcal{B})=(n+1)^2$. Although there are many such bases in G, they have the same structure as we have shown in Theorem J. Since our definition of non-separating cycles on locally orientable surface refuses the existence of facial cycles in such shortest non-separating cycle base, there may exist an edge contained in exactly one cycle in such a base. For instance, the edge (x_1,y_n) in Fig.1 is contained in exactly one non-separating cycle of such shortest cycle base.

References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications* , Macmillan, London, 1978.

[2] A.C. Casell et al., Cycle bases of minimum measure for the structural analysis of skeletal structures by the flexibility method, *Proc. R. Soc. London Ser. A* 35 (1976) 61-70.

[3] G.D. Downs et al., Review of ring perception algorithms for chemical graphs, *J. Chem. Inf. Comput. Sci.* 29 (1989) 172-187.

[3] S.M. Freier et al., Improved free-energy parameters for predictions of RNA duplex stability, *Proc. Natl. Acad. Sci.* (USA) 83 (1986) 9373-9377.

[4] P. Hall, On representatives of subsets, *London Math. Soc.* 10 (1935) 26-30.

[5] L.F.Mao, A new view of combinatorial maps by Smarandache's notion, in *Selected Papers on Mathematical Combinatorics*(I), World Academic Union, 2006, also in *arXiv: math.GM/0506232*.

[6] L.F.Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[7] J.B. Mohar and C. Thomassen, *Graphs on Surfaces* , The Johns Hopkins University Press, Baltimore and London, 2001.

[8] W. Tutte, How to draw a graph, *Proc. London Math. Soc.* 13 (1963) 743-768.

[9] W. Tutte, A homotopy theorem for matroids I, II, *Trans. AMS* 88 (1958) 144-160, 161-174.

[10] C.J.A. Welsh, *Matroid Theory*, Academic Press, 1976.

[11] A.L. White, *Theory of Matroids*, Cambridge University Press, 1986.

[12] H. Whitney, On abstract properties of linear dependence, *Amer. J. Math.* 57 (1953) 509-533.

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The Crossing Number of $K_{1,5,n}$

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Abstract: In this paper, we determine the crossing number of the complete tripartite graph $K_{1,5,n}$ for any integer $n \geq 1$, related with Smarandache 2-manifolds on spheres.

Key Words: Good drawings, complete tripartite graphs, crossing number.

AMS(2000) : O5C10

§1. Introduction

The *crossing number* $cr(G)$ of a graph G is the smallest crossing number among all drawings of G in the plane. It is well known that the crossing number of graph is attained only in good drawings of the graph related with *map geometries*, i.e., *Smarandache* 2*-manifolds* (see [8] for details), which are those drawings where no edges cross itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges intersect in a common point. Let ϕ be a good drawing of the graph G, we denote the number of crossings in this drawing of G by cr_{ϕ}(G).

The investigation on the crossing number of a graph is a classical and however very difficult problem (for example, see [3]). Garey and Johnson [4] have proved that the problem to determine the crossing number of a graph is NP-complete. Because of its difficulty, presently we only know the crossing number of some classes of special graphs, for example: the complete graphs with small number of vertices ([15]), the complete bipartite graph of less number of vertices in one bipartite partition ([7],[15]), certain generalized Peterson graphs ([12]), and some Cartesian product graphs of two circuits([2],[11]-[14]), of path and stars ([9]).

The crossing numbers of complete bipartite graphs $K_{m,n}$ were computed by D.J.Kleitman [7], for the case $m \leq 6$. He proved that

$$cr(K_{m,n}) = Z(m,n), \text{if } m \leq 6, \quad \text{where} \quad Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor.$$

On the crossing number of the complete tripartite graphs, as far as the authors know, there

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only are the following two results: Kouhei Asano [1] proved that

$$cr(K_{1,3,n}) = Z(4,n) + \lfloor \frac{n}{2} \rfloor, \text{and } cr(K_{2,3,n}) = Z(5,n) + n;$$

and Huang [5] recently proves that $cr(K_{1,4,n}) = n(n-1)$.

In this paper, using Kleitman's theorem, we determine the crossing number of complete tripartite graph $K_{1,5,n}$ for any integer $n \geq 1$. The main result of this paper is the following theorem.

Theorem 1 (the main result) *For any integer $n \geq 1$,*

$$cr(K_{1,5,n}) = Z(6,n) + 4\lfloor \frac{n}{2} \rfloor$$

We now explain some notations. Let G be a graph with vertex set V and edge set E. If $A \subseteq E$ (or $A \subseteq V$), we use $G\langle A\rangle$ to denote the subgraph of G induced by A; if G is known from the context, we simply write $\langle A\rangle$ instead of $G\langle A\rangle$. For two mutually disjoint subsets X and Y of V, we use E_{XY} to denote all the edges of G incident with a vertex in X and a vertex in Y. For a vertex v, E_v denotes all the edges of G incident with v.

Let A and B be two sets of edges of a graph G. If ϕ is a good drawing of G, we denote $cr_\phi(A,B)$ by the number of all crossings whose two crossed edges are respectively in A and in B. Especially, $cr_\phi(A,A)$ will be denoted by $cr_\phi(A)$. If G has the edge set E, the two signs $cr_\phi(G)$ and $cr_\phi(E)$ are essential the same.

The following formulas, which can be shown easily, are usually used in the proofs of our lemmas and theorem.

$$\begin{aligned} cr_\phi(A \cup B) &= cr_\phi(A) + cr_\phi(B) + cr_\phi(A,B) \\ cr_\phi(A, B \cup C) &= cr_\phi(A,B) + cr_\phi(A,C), \end{aligned} \tag{1}$$

where A, B and C are mutually disjoint subsets of E.

In the next section we shall give some lemmas, and then prove our theorem in the last one.

§2. Some Lemmas

Lemma 2.1 *Let G be a complete bipartite graph $K_{m,n}$ with the edge set E and the vertex bipartition (Y,Z), where $Y = \{y_1, \cdots, y_m\}$, and $Z = \{z_1, \cdots, z_n\}$. If ϕ is any good drawing of G, then*

$$(n-2)cr_\phi(E) = \sum_{i=1}^{n} cr_\phi(E \setminus E_{z_i}).$$

Proof The conclusion follows from the fact that in the drawing of $K_{m,n}$, there are n drawings of $K_{m,n-1}$, and each crossing occurs in $(n-2)$ of them. □

Lemma 2.2 *Let G be a complete tripartite graph $K_{s,m,n}$ with the edge set E and the vertex tripartition (X,Y,Z), where $X = \{x_1, \cdots, x_s\}$, $Y = \{y_1, \cdots, y_m\}$, and $Z = \{z_1, \cdots, z_n\}$. If ϕ is any good drawing of G, then we have*

(i) $\sum_{i=1}^{n} cr_\phi(E \setminus E_{z_i}) = (n-2)cr_\phi(E) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) + 2cr_\phi(E_{XY})$;

(ii) $\sum_{i=1}^{m} cr_\phi(E \setminus E_{y_i}) = (m-2)cr_\phi(E) + \sum_{i=1}^{m} cr_\phi(E_{XZ}, E_{y_i}) + 2cr_\phi(E_{XZ})$

Proof We only prove (i), because (ii) is analogous by the symmetry of the vertex tripartition of G. Using the formula (1), we have

$$\begin{aligned} cr_\phi(E) &= cr_\phi(E_{XY} \cup E_{XZ} \cup E_{YZ}) \\ &= cr_\phi(E_{XY}) + cr_\phi(E_{XZ} \cup E_{YZ}) + cr_\phi(E_{XY}, E_{XZ} \cup E_{YZ}) \\ &= cr_\phi(E_{XY}) + cr_\phi(E_{XZ} \cup E_{YZ}) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) \end{aligned} \tag{2}$$

Since $\langle E_{XZ} \cup E_{YZ} \rangle$ is isomorphic to the complete bipartite graph $K_{s+m,n}$ with the vertex bipartition $(X \cup Y, Z)$, it follows from by Lemma 2.1 that

$$(n-2)cr_\phi(E_{XY} \cup E_{YZ}) = \sum_{i=1}^{n} cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) \tag{3}$$

On the other hand, using the formula (1) again we have

$$\begin{aligned} cr_\phi(E \setminus E_{z_i}) &= cr_\phi\Big((E_{XY} \cup E_{XZ} \cup E_{YZ}\Big) \setminus E_{z_i}\Big) \\ &= cr_\phi(E_{XY}) + cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E(z_i)\Big) \\ &\quad + cr_\phi\Big(E_{XY}, (E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) \\ &= cr_\phi(E_{XY}) + cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) \\ &\quad + \sum_{j=1}^{n} cr_\phi(E_{XY}, E_{z_j}) - cr_\phi(E_{XY}, E_{z_i}), \end{aligned}$$

namely, we have

$$cr_\phi(E \setminus E_{z_i}) = cr_\phi(E_{XY}) + cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) + \sum_{j=1}^{n} cr_\phi(E_{XY}, E_{z_j}) - cr_\phi(E_{XY}, E_{z_i}) \tag{4}$$

Taking sum for i on the two sides of (4) above, we obtain that

$$\begin{aligned}
\sum_{i=1}^{n} cr_\phi(E \setminus E_{z_i}) &= ncr_\phi(E_{XY}) + \sum_{i=1}^{n} cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) \\
&\quad + \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} cr_\phi(E_{XY}, E_{z_j}) - cr_\phi(E_{XY}, E_{z_i})\Big) \\
&= ncr_\phi(E_{XY}) + \sum_{i=1}^{n} cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) + (n-1)\sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) \\
&= ncr_\phi(E_{XY}) + (n-2)cr_\phi(E_{XY} \cup E_{YZ}) \\
&\quad +(n-1)\sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) \qquad \text{(by (3) above)} \\
&= 2cr_\phi(E_{XY}) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) \\
&\quad +(n-2)\Big(cr_\phi(E_{XY}) + cr_\phi(E_{XZ} \cup E_{YZ}) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i})\Big) \\
&= 2cr_\phi(E_{XY}) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) + (n-2)cr_\phi(E) \qquad \text{(by (2) above)}
\end{aligned}$$

This proves the lemma. □

Note that in Lemma 2 above, if X is a set containing a single vertex x, then E_{XY} is the set of edges incident to x, and thus $cr_\phi(E_{XY}) = 0$ by any good drawing ϕ.

Lemma 2.3 *Let G be a complete tripartite graph $K_{1,5,n}$ with the edge set E and the vertex tripartition (X, Y, Z), where $X = \{x\}$, $Y = \{y_1, \cdots, y_5\}$, and $Z = \{z_1, \cdots, z_n\}$. If ϕ is a good drawing of G satisfying that $cr_\phi(E) = Z(6, n) + 4\left[\frac{n}{2}\right] - a$ for some a. Then we have*

(1) if $n = 2k$, then $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) \geq 2k^2 - 2k + 3a$;

(2) if $n = 2k + 1$, then $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) \geq 2k^2 - 4 + 3a$.

Proof Let e_i denote the edge xy_i for $1 \leq i \leq 5$, and f_j denote the edge xz_j for $1 \leq j \leq n$. Without loss of generality, assume that under the drawing ϕ, the reverse clock order of these five edges e_i $(1 \leq i \leq 5)$ around x is: $e_1 \to e_2 \to e_3 \to e_4 \to e_5$. These five edges form five angles: $\alpha_i = \angle e_i x e_{i+1}$, where $1 \leq i \leq 5$ and the indices are read module 5. We see that in the plane R^2, there exists a circle neighbor $N(x, \varepsilon) = \{s \in R^2 : ||s - x|| < \varepsilon\}$, where ε is a sufficiently small positive number, such that for any other edge e of $K_{1,5,n}$ not incident with x, e can not be located in $N(x, \varepsilon)$. Since the graph $K_{1,5,n}$ has still n edges f_j that are incident to x $(1 \leq j \leq n)$, let A_i denote the set of all those edges f_j, each of which lies in the angle α_i (see the Fig. 1 in the next page). Clearly, we have that $|A_1| + |A_2| + |A_3| + |A_4| = n$.

In the following, associated with the drawing ϕ of G, we shall produce five new graphs G_i, together with their respective good drawing ϕ_i $(1 \leq i \leq 5)$, where each G_i is isomorphic to the complete bipartite graph $K_{5,n+1}$. We shall heavily illustrate how to obtain the graph G_1 and its drawing ϕ_1, for the rest cases the method is analogous.

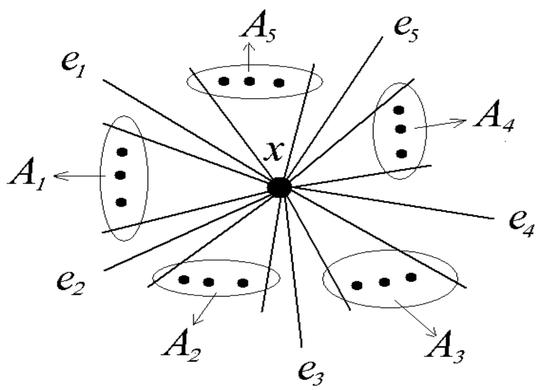

Fig. 1

First, we delete all edges in E_{y_1} and the vertex y_1 from G, and then remove the part of e_i lying in $N(x,\varepsilon)$ for $2 \le i \le 5$ (not remove the vertex x); add a new vertex z_{n+1} in some location of $e_4 \cap N(x,\varepsilon)$. Now we connected z_{n+1} to x and y_i $(i = 2,3,4,5)$ by the following way: connect z_{n+1} to x and y_4 respectively along the original two sections of e_4; connect z_{n+1} to y_3 by first traversing through α_3 (near to x) and then along the original section of e_3 lying out $N(x,\varepsilon)$; connect z_{n+1} to y_2 by successively traversing through α_3 and α_2 (near to x) and then along the original section of e_2 lying out $N(x,\varepsilon)$; connect z_{n+1} to y_5 by first traversing through α_4 (near to x) and then along the original section of e_5 lying out $N(x,\varepsilon)$. Then we obtain the graph G_1 with its a good drawing ϕ_1. Obviously, G_1 is isomorphic to $K_{5,n+1}$. The following figure 2 helps us to understand the obtained graph G_1 and its drawing ϕ_1, where the dotted line denote the way how z_{n+1} is connected to x and y_i $(2 \le i \le 5)$.

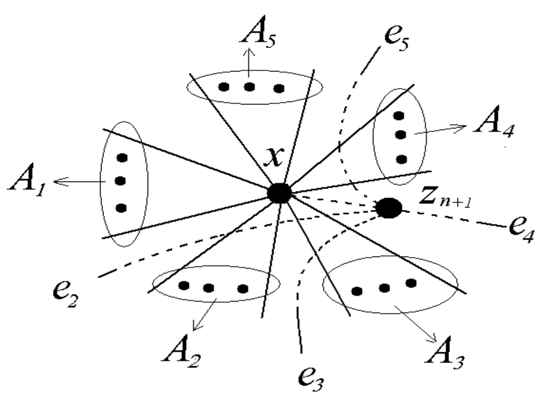

Fig. 2

Then it is not difficult to see that

$$cr_{\phi_1}(G_1) = cr_{\phi}\Big(E \setminus E_{y_1}\Big) + |A_2| + 2|A_3| + |A_4|. \tag{4}$$

By the symmetry of y_i, we can analogously easily obtain the graphs G_i and its goods drawings ϕ_i for $2 \le i \le 5$. For example, the graph G_2, together with its good drawing ϕ_2, is displayed in

the following figure 3.

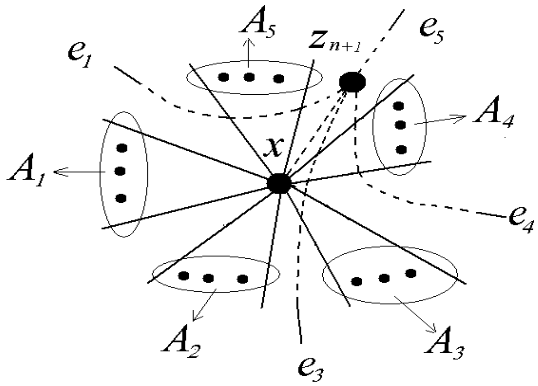

Fig. 3

Similarly, for ϕ_2, ϕ_3, ϕ_4 and ϕ_5, we have respectively the following equalities :

$$cr_{\phi_2}(G_2) = cr_{\phi}(E \setminus E_{y_2}) + |A_3| + 2|A_4| + |A_5| \tag{5}$$

$$cr_{\phi_3}(G_3) = cr_{\phi}(E \setminus E_{y_3}) + |A_1| + 2|A_5| + |A_4| \tag{6}$$

$$cr_{\phi_4}(G_4) = cr_{\phi}(E \setminus E_{y_4}) + |A_2| + 2|A_1| + |A_5| \tag{7}$$

$$cr_{\phi_5}(G_5) = cr_{\phi}(E \setminus E_{y_5}) + |A_1| + 2|A_2| + |A_3| \tag{8}$$

Since each G_i $(1 \leq i \leq 5)$ is isomorphic to the complete graph $K_{5,n+1}$, we get that $cr_{\phi_i}(G_i) \geq Z(5, n+1)$. Therefore, by (4)–(8) above, we have

$$\begin{aligned}
5Z(5,n+1) &\leq \sum_{i=1}^{5} cr_{\phi_i}(G_i) \\
&= \sum_{i=1}^{5} cr_{\phi}(E \setminus E_{y_i}) + 4\sum_{i=1}^{5} |A_i| \\
&= \sum_{i=1}^{5} cr_{\phi}(E \setminus E_{y_i}) + 4n \\
&= 3cr_{\phi}(E) + \sum_{i=1}^{5} cr_{\phi}(E_{XZ}, E_{y_i}) + 2cr_{\phi}(E_{XY}) + 4n \quad \text{(by Lemma \textbf{??} (2))} \\
&= 3cr_{\phi}(E) + \sum_{i=1}^{5} cr_{\phi}(E_{XZ}, E_{y_i}) + 4n \quad \text{(because } cr_{\phi}(E_{XY}) = 0\text{)}
\end{aligned}$$

So it follows that

$$\begin{aligned}\sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) &\geq 5Z(5,n+1) - 3cr_\phi(E) - 4n\\ &= 5Z(5,n+1) - 3\Big(Z(6,n) + 4\left[\frac{n}{2}\right] - a\Big) - 4n\\ &= \begin{cases} 2k^2 - 2k + 3a, & \text{when } n = 2k;\\ \\ 2k^2 - 4 + 3a, & \text{when } n = 2k+1 \end{cases}\end{aligned}$$

This proves the lemma. □

Lemma 2.4 *Let G be the complete tripartite graph $K_{1,5,n}$ with the edge set E and the vertex tripartition (X, Y, Z), where $X = \{x\}$, $Y = \{y_1, \cdots, y_5\}$, and $Z = \{z_1, \cdots, z_n\}$. Assume that $n = 2k + 1$, where $k \geq 0$. If ϕ is a good drawing of G satisfying that $cr_\phi(E \setminus E_{z_j}) = Z(6, n-1) + 4\left[\frac{n-1}{2}\right]$ for any $1 \leq j \leq n$, then $cr_\phi(E) \neq Z(6,n) + 4\left[\frac{n}{2}\right] - 1$.*

Proof Assume to contrary that $cr_\phi(E) = Z(6,n) + 4\left[\frac{n}{2}\right] - 1$. By using the formula (2) in the proof of lemma 2.2, we have

$$Z(6,n) + 4\left[\frac{n}{2}\right] - 1 = cr_\phi(E) = cr_\phi(E_{XZ}) + cr_\phi(E_{XY} \cup E_{YZ}) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}).$$

Since $\langle E_{XY} \cup E_{YZ}\rangle$ is isomorphic to the complete bipartite graph $K_{5,n+1}$, we have that $cr_\phi(E_{XY} \cup E_{YZ}) \geq Z(5, n+1)$. Noting that $cr_\phi(E_{XZ}) = 0$, we thus have

$$\sum_{i=1}^{5} cr_\phi(E_{XY}, E_{y_i}) \leq Z(6,n) + 4\left[\frac{n}{2}\right] - 1 - Z(5, n+1) = 2k^2 - 1$$

On the other hand, by our assumption that $cr_\phi(E) = Z(6,n)+4\left[\frac{n}{2}\right]-1$, and that $n = 2k+1$, with the help of Lemma 2.3(ii) we have $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) \geq 2k^2 - 1$. This implies that

$$\sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) = 2k^2 - 1 \tag{9}$$

Since $\langle E \setminus E_{z_j}\rangle$ is isomorphic to the complete tripartite graph $K_{1,m,n-1}$ with the vertex tripartition $(X, Y, Z \setminus \{z_j\})$, applying the formula (2) in the proof of Lemma 2.2 to the graph $\langle E \setminus E_{z_j}\rangle$, we have

$$cr_\phi(E \setminus E_{z_j}) = cr_\phi\Big(E_{X(Z\setminus\{z_j\})}\Big) + cr_\phi\Big(E_{XY} \cup E_{Y(Z\setminus\{z_j\})}\Big) + \sum_{i=1}^{5} cr_\phi\Big(E_{X(Z\setminus\{z_j\})}, E'_{y_i}\Big),$$

where $E'_{y_i} = E_{X\{y_i\}} \cup E_{(Z\setminus\{z_j\})\{y_i\}}$.

Since $\langle E_{XY} \cup E_{Y(Z\setminus\{z_j\})}\rangle$ is isomorphic to the complete bipartite graph $K_{5,n}$, $cr_\phi\Big(E_{XY} \cup E_{Y(Z\setminus\{z_j\})}\Big) \geq Z(5,n)$. Again, since $E_{X(Z\setminus\{z_j\})}$ is the set of edges incident to x, we have that

$cr_\phi\Big(E_{X(Z\setminus\{z_j\})}\Big) = 0$ by the good drawing ϕ. Therefore we have

$$\begin{aligned}\sum_{i=1}^{5} cr_\phi\Big(E_{X(Z\setminus\{z_j\})}, E'_{y_i}\Big) &= cr_\phi(E\setminus E_{z_j}) - cr_\phi\Big(E_{X(Z\setminus\{z_j\})}\Big) - cr_\phi\Big(E_{XY}\cup E_{Y(Z\setminus\{z_j\})}\Big)\\ &= cr_\phi(E\setminus E_{z_j}) - cr_\phi\Big(E_{XY}\cup E_{Y(Z\setminus\{z_j\})}\Big)\\ &\leq Z(6,n-1) + 4\left[\frac{n-1}{2}\right] - Z(5,n)\\ &= 2k^2 - 2k\end{aligned}$$

That is to say, we have

$$\sum_{i=1}^{5} cr_\phi\Big(E_{X(Z\setminus\{z_j\})}, E'_{y_i}\Big) \leq 2k^2 - 2k \tag{10}$$

Because $E_{X\{z_j\}} \cup E_{\{z_j\}\{y_i\}}$ is the set of edges incident to z_j, $cr_\phi(E_{X\{z_j\}}, E_{\{z_j\}\{y_i\}}) = 0$ by the good drawing ϕ. Note that $E'_{y_i} = E_{y_i}\setminus E_{\{z_j\}\{y_i\}}$. Hence, we have

$$\begin{aligned}cr_\phi(E_{XZ}, E_{y_i}) &= cr_\phi(E_{XZ}, E'_{y_i}) + cr_\phi(E_{X(Z\setminus\{z_j\})}, E_{\{z_j\}\{y_i\}})\\ &= \Big(cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}, E'_{y_i})\Big) + cr_\phi(E_{X(Z\setminus\{z_j\})}, E_{\{z_j\}\{y_i\}})\\ &= cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}, E_{y_i}\setminus E_{\{z_j\}\{y_i\}})\\ &\quad + cr_\phi(E_{X(Z\setminus\{z_j\})}, E_{\{z_j\}\{y_i\}})\\ &= cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}, E_{y_i}) - cr_\phi(E_{X\{z_j\}}, E_{\{z_j\}\{y_i\}})\\ &\quad + cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}}) - cr_\phi(E_{X\{z_j\}}, E_{\{z_j\}\{y_i\}})\\ &= cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}), E_{y_i}) + cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}})\end{aligned}$$

Taking sum for i on two sides of the last equality above, we have

$$\begin{aligned}\sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) &= \sum_{i=1}^{5} cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + \sum_{i=1}^{5} cr_\phi(E_{X\{z_j\}}, E_{y_i})\\ &\quad + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}})\end{aligned}$$

Combining with (9) and (10) above, we then obtain that

$$2k^2 - 1 \leq 2k^2 - 2k + \sum_{i=1}^{5} cr_\phi(E_{X\{z_j\}}, E_{y_i}) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}}) \tag{11}$$

Again, taking sum for j on the two sides of the inequality (11) above, and noticing $n = 2k+1$, we get that

$$\begin{aligned}
\sum_{j=1}^{n}(2k^2-1) &\leq \sum_{j=1}^{n}(2k^2-2k)+\sum_{j=1}^{n}\sum_{i=1}^{5}cr_\phi(E_{X\{z_j\}},E_{y_i})+\sum_{j=1}^{n}\sum_{i=1}^{5}cr_\phi(E_{XZ},E_{\{z_j\}\{y_i\}})\\
&= (2k+1)(2k^2-2k)+\sum_{i=1}^{5}\Big(\sum_{j=1}^{n}cr_\phi(E_{X\{z_j\}},E_{y_i})\Big)\\
&\quad +\sum_{i=1}^{5}\Big(\sum_{j=1}^{n}cr_\phi(E_{XZ},E_{\{z_j\}\{y_i\}})\Big)\\
&= (2k+1)(2k^2-2k)+\sum_{i=1}^{5}cr_\phi(E_{XZ},E_{y_i})+\sum_{i=1}^{5}cr_\phi(E_{XZ},E_{Z\{y_i\}})\\
&= (2k+1)(2k^2-2k)+\sum_{i=1}^{5}cr_\phi(E_{XZ},E_{y_i})\\
&\quad +\sum_{i=1}^{5}\Big(cr_\phi(E_{XZ},E_{y_i})-cr_\phi(E_{XZ},E_{X\{y_i\}}\Big) \qquad (\text{because } E_{Z\{y_i\}}=E_{y_i}\setminus E_{X\{y_i\}}\,)\\
&= (2k+1)(2k^2-2k)+2\sum_{i=1}^{5}cr_\phi(E_{XZ},E_{y_i})
\end{aligned}$$

(This is because $E_{XZ}\cup E_{X\{y_i\}}$ is the set of edges incident to x, by the good drawing ϕ , $cr_\phi(E_{XZ},E_{Z\{y_i\}})=0$ for any $1\leq i\leq 5$)

$$= (2k+1)(2k^2-2k)+2(2k^2-1) \qquad (\text{ by (9) above})$$

Therefore, it follows that $(2k+1)(2k-1)\leq 2(2k^2-1)$. This is a contradiction for any real number k, and proving the conclusion. □

§3. Proof of Theorem 1

Let the complete tripartite graph $K_{1,5,n}$ having the edge set E and the vertex tripartition (X,Y,Z), where $X=\{x\}$, $Y=\{y_1,\cdots,y_5\}$, and $Z=\{z_1,\cdots,z_n\}$. To show that $cr(K_{1,5,n})\leq Z(6,n)+4\left[\frac{n}{2}\right]$, we consider a drawing of $K_{1,5,n}$ as a immersion into R^2, satisfying the following:

(1) $\phi(x)=(0,1)$;

(2) $\phi(y_i)=(0,(-1)^i i), i=1,2, \phi(y_3)=(\varepsilon,-2), \phi(y_4)=(\varepsilon,3), \phi(y_5)=(2\varepsilon,4)$,where ε is a sufficiently small positive;

(3) $\phi(z_j)=\left((-1)^j\left[\frac{j+1}{2}\right],0\right)$.

For example, a drawing of $K_{1,5,5}$ on the plane is shown in the Fig.4. It is not difficult to see that $cr_\phi(E)=Z(6,n)+4\left[\frac{n}{2}\right]$. This thus shows that $cr(K_{1,5,n})\leq Z(6,n)+4\left[\frac{n}{2}\right]$. In order to prove the theorem, we only need to prove the conclusion that $cr_\phi(K_{1,5,n})\geq Z(6,n)+4\left[\frac{n}{2}\right]$ for any good drawing ϕ. Assume to contrary that there is a good drawing ϕ of $K_{1,5,n}$ satisfying $cr_\phi(K_{1,5,n})=Z(6,n)+4\left[\frac{n}{2}\right]-a$, where $a\geq 1$. We now consider the following two cases, according to as n is even or odd.

Claim 1 *The desired conclusion is true when $n\,(=2k)$ is even.*

Subproof By our assumption that $cr_\phi(K_{1,5,n}) = Z(6,n) + 4\left[\frac{n}{2}\right] - a$, it then follows from Lemma 2.3(i) that

$$\sum_{i=1}^{5} cr_\phi(E_{xz}, E_{y_i}) \geq 2k^2 - 2k + 3a.$$

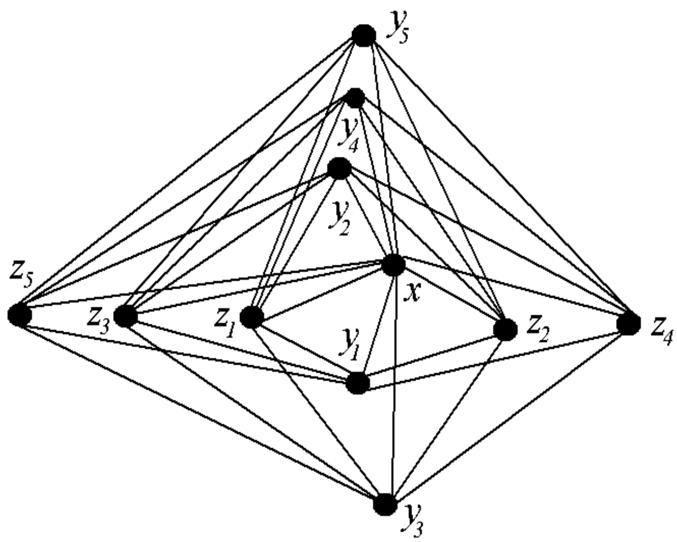

Fig. 4

Note that $cr_\phi(E_{XZ}) = 0$ by the good drawing ϕ. Since $\langle E_{XZ} \cup E_{YZ}\rangle$ is isomorphic to the complete bipartite graph $K_{5,n+1}$ with the vertex bipartition $(Y, X \cup Z)$, we have that $cr_\phi(E_{XY} \cup E_{YZ}) \geq Z(5, n+1)$. Using the formulas (2) in the proof of Lemma 2.2, we get that

$$\begin{aligned} Z(6,n) + 4\left[\frac{n}{2}\right] - a &= cr_\phi(E) \\ &= cr_\phi(E_{XZ}) + cr_\phi(E_{XY} \cup E_{YZ}) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) \\ &\geq Z(5, n+1) + \sum_{i=1}^{5}(E_{XZ}, E_{y_i}) \end{aligned}$$

Therefore, $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ}, E(y_i)) \leq 2k^2 - 2k - a$. So, we get that $2k^2 - 2k + 3a \leq 2k^2 - 2k - a$, namely, $a \leq 0$. This contradicts to the hypothesis that $a \geq 1$, proving the claim.

Claim 2 *The desired conclusion is true when $n\,(= 2k+1)$ is odd.*

Subproof. Since n is odd, by Lemma 2.3(ii) we first have

$$\sum_{i=1}^{5} cr_\phi(E_{XZ}, E(y_i)) \geq 2k^2 - 4 + 3a$$

Similarly, using the formulas (2) in the proof of Lemma 2.2, we get that

$$\begin{aligned} Z(6,n)+4\left[\frac{n}{2}\right]-a &= cr_\phi(E) \\ &\geq Z(5,n+1)+\sum_{i=1}^{5} cr_\phi(E_{XZ},E_{y_i}), \end{aligned}$$

which follows that $\sum_{i=1}^{5}(E_{XZ},E_{y_i}) \leq 2k^2-a$. Hence, we get that $2k^2-4+3a \leq 2k^2-a$, namely $a \leq 1$. Since $a \geq 1$ by our assumption, this implies that $a=1$, and thus it must be that

$$cr_\phi(E) = Z(6,n)+4\left[\frac{n}{2}\right]-1 \tag{12}$$

Again, with the help of the formula (1), we have

$$\begin{aligned} cr_\phi(E) &= cr_\phi(E_{XY}\cup E_{XZ}\cup E_{YZ}) \\ &= cr_\phi(E_{XY})+cr_\phi(E_{XZ}\cup E_{YZ})+cr_\phi(E_{XY},E_{XZ}\cup E_{YZ}) \\ &= cr_\phi(E_{XY})+cr_\phi(E_{XZ}\cup E_{YZ})+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i}) \end{aligned}$$

Since $\langle E_{XZ}\cup E_{YZ}\rangle$ is isomorphic to the complete bipartite graph $K_{6,n}$ with the vertex bipartition $(X\cup Y, Z)$, it has that $cr_\phi(E_{XZ}\cup E_{YZ}) \geq Z(6,n)$. Noting that $cr_\phi(E_{XY})=0$ by the good drawing of ϕ, we thus have

$$Z(6,n)+4\left[\frac{n}{2}\right]-1 = cr_\phi(E) \geq Z(6,n)+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i}),$$

which follows that

$$\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i}) \leq Z(6,n)+4\left[\frac{n}{2}\right]-1-Z(6,n) = 4k-1 \tag{13}$$

Combining with Lemma 2.2(i), we have

$$\begin{aligned} \sum_{i=1}^{n} cr_\phi(E\setminus E_{z_i}) &= 2cr_\phi(E_{XY})+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_j})+(n-2)cr_\phi(E) \\ &= \sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_j})+(n-2)cr_\phi(E) \quad (\text{because } cr_\phi(E_{XY})=0) \\ &\leq 4k-1+(n-2)\Big(Z(6,n)+4\left[\frac{n}{2}\right]-1\Big) \quad (\text{by (12) and (13) above}) \\ &= n\Big(Z(6,n-1)+4\left[\frac{n-1}{2}\right]\Big) \qquad (\text{because } n=2k+1\ ) \end{aligned}$$

That is to say, we have

$$\sum_{i=1}^{n} cr_\phi(E\setminus E_{z_i}) \leq n\Big(Z(6,n-1)+4\left[\frac{n-1}{2}\right]\Big) \tag{14}$$

On the other hand, since, for any $1 \leq i \leq n$, $\langle E \setminus E(z_i)\rangle$ is isomorphic to the complete tripartite graph $K_{1,5,n-1}$, and since $n-1$ is even, it follows from the truth of Claim 1 that $cr_\phi(E \setminus E_{z_i}) \geq Z(6, n-1) + 4\left[\frac{n-1}{2}\right]$ for any $1 \leq i \leq n$. Combined with (14) above, it must happen that $cr_\phi(E \setminus E_{z_i}) = Z(6, n-1) + 4\left[\frac{n-1}{2}\right]$ for any $1 \leq i \leq n$. This, together with n being odd and (12) above, contradicts Lemma 2.4, proving this claim.

Therefore, the proof of Theorem 1 is finished. □

References

[1] Kouchei Asano, The crossing number of $K_{1,3,n}$ and $k_{2,3,n}$, *J.Graph Theory*, **10** (1980), 1-8.

[2] A.M.Dean and R.B.Richter, The crossing number of $C_4 \times C_4$, *J.Graph Theory*, **19** (1995), 125-129.

[3] P.Erdös and R.K.Guy, Crossing number problems, *Am.Math.Month*, **80** (1973), 52-58.

[4] M.R.Garey and D.S.Johnson, Crossing number is NP-complete, *SIAM J.Algebraic.Discrete Methods*, **4** (1983), 312-316.

[5] Yuanqiu Huang, The crossing number of $K_{1,4,n}$, *Discrete Mathematics*(to appear).

[6] Stanislav Jendrol and Marián Klešč, On the graphs whose line graphs have crossing number one , *J. Graph Theory*, **37** (2001), 181-188.

[7] D.J.Kleitman, The crossing number of $K_{5,n}$, *J.Combinatorial Theory*, **9** (1970) 315-323.

[8] L.F.Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[9] Marián Klešč, The crossing numbers of products of path ans stars with 4-vertex graphs,*J. Graph Theory*, **18**(6) (1994), 605-614.

[10] V.R.Kulli, D.G.Akka and L.W.Beineke, On the line graphs with crossing number 1, *J. Graph Theory*, **3** (1979), 87-90.

[11] Marián Klešč,R.Bruce Richter and Ian Stobert, The crossing number of $C_5 \times C_n$, *J.Graph Theory*, **22**(3) (1996), 239-243.

[12] Dan McQuillan and R.Bruce Richter, On the crossing numbers of certain generalized peterson graphs, *Discrete Math.*, **104** (1992), 311-320.

[13] R.Bruce Richter and Gelasio Salazar, The crossing number of $C_6 \times C_n$, *Australasian J.of Combinatorics*, **23** (2001), 135-143.

[14] R.B.Richter and C.Thomassen, Intersection of curve systems and the crossing number of $C_5 \times C_5$,*Discrete Comp.Geom.*, **13**(1995), 149-159.

[15] D.R.Woodall, Cyclic-order graphs and Zarankiewicz's crossing number conjecture *J.Graph Theory*, **17**(1993), 657-671.

Math.Combin.Book.Ser. Vol.1 (2007), 45-58

Pseudo-Manifold Geometries with Applications

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Abstract: A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways and a Smarandache n-manifold is a n-manifold that support a Smarandache geometry. Iseri provided a construction for Smarandache 2-manifolds by equilateral triangular disks on a plane and a more general way for Smarandache 2-manifolds on surfaces, called *map geometries* was presented by the author in [9] − [10] and [12]. However, few observations for cases of $n \geq 3$ are found on the journals. As a kind of Smarandache geometries, a general way for constructing dimensional n pseudo-manifolds are presented for any integer $n \geq 2$ in this paper. Connection and principal fiber bundles are also defined on these manifolds. Following these constructions, nearly all existent geometries, such as those of *Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kähler geometry* and *Finsler geometry*, ...,etc., are their sub-geometries.

Key Words: Smarandache geometry, Smarandache manifold, pseudo-manifold, pseudo-manifold geometry, multi-manifold geometry, connection, curvature, Finsler geometry, Riemann geometry, Weyl geometry and Kähler geometry.

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§1. Introduction

Various geometries are encountered in update mathematics, such as those of *Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kähler geometry* and *Finsler geometry*, ..., etc.. As a branch of geometry, each of them has been a kind of spacetimes in physics once and contributes successively to increase human's cognitive ability on the natural world. Motivated by a combinatorial notion for sciences: *combining different fields into a unifying field*, Smarandache introduced *neutrosophy and neutrosophic logic* in references [14] − [15] and Smarandache geometries in [16].

Definition 1.1([8][16]) *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache geometry is a geometry which has at least one Smarandachely denied

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*axiom(*1969*).*

Definition 1.2 *For an integer $n, n \geq 2$, a Smarandache n-manifold is a n-manifold that support a Smarandache geometry.*

Smarandache geometries were applied to construct many world from conservation laws as a mathematical tool([2]). For Smarandache n-manifolds, Iseri constructed Smarandache manifolds for $n = 2$ by equilateral triangular disks on a plane in [6] and [7] (see also [11] in details). For generalizing Iseri's Smarandache manifolds, *map geometries* were introduced in $[9] - [10]$ and [12], particularly in [12] convinced us that these map geometries are really Smarandache 2-manifolds. Kuciuk and Antholy gave a popular and easily understanding example on an Euclid plane in [8]. Notice that in [13], these *multi-metric space* were defined, which can be also seen as Smarandache geometries. However, few observations for cases of $n \geq 3$ and their relations with existent manifolds in differential geometry are found on the journals. The main purpose of this paper is to give general ways for constructing dimensional n pseudo-manifolds for any integer $n \geq 2$. Differential structure, connection and principal fiber bundles are also introduced on these manifolds. Following these constructions, nearly all existent geometries, such as those of *Euclid geometry, Lobachevshy-Bolyai geometry, Riemann geometry, Weyl geometry, Kähler geometry* and *Finsler geometry*, ...,etc., are their sub-geometries.

Terminology and notations are standard used in this paper. Other terminology and notations not defined here can be found in these references $[1], [3] - [5]$.

For any integer $n, n \geq 1$, an *n-manifold* is a Hausdorff space M^n, i.e., a space that satisfies the T_2 separation axiom, such that for $\forall p \in M^n$, there is an open neighborhood $U_p, p \in U_p \subset M^n$ and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$ or $\mathbf{C}^n$, respectively.

Considering the differentiability of the homeomorphism $\varphi : U \to \mathbf{R}^n$ enables us to get the conception of differential manifolds, introduced in the following.

An *differential n-manifold* $(M^n, \mathcal{A})$ is an n-manifold $M^n, M^n = \bigcup\limits_{i \in I} U_i$, endowed with a C^r differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n for an integer r with following conditions hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of M^n;

(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\beta(U_\beta) \ \text{ and } \ \varphi_\beta \varphi_\alpha^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\alpha(U_\alpha)$$

are C^r;

(3) $\mathcal{A}$ is maximal, i.e., if (U, φ) is an atlas of M^n equivalent with one atlas in $\mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

An n-manifold is *smooth* if it is endowed with a C^∞ differential structure. It is well-known that a complex manifold M^n_c is equal to a smooth real manifold M^{2n}_r with a natural base

$$\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} | \ 1 \leq i \leq n\}$$

for $T_p M^n_c$, where $T_p M^n_c$ denotes the tangent vector space of M^n_c at each point $p \in M^n_c$.

§2. Pseudo-Manifolds

These Smarandache manifolds are non-homogenous spaces, i.e., there are singular or inflection points in these spaces and hence can be used to characterize warped spaces in physics. A generalization of ideas in map geometries can be applied for constructing dimensional n pseudo-manifolds.

Construction 2.1 *Let M^n be an n-manifold with an atlas $\mathcal{A} = \{(U_p, \varphi_p)|p \in M^n\}$. For $\forall p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, define a spatially directional mapping $\omega : p \to \mathbf{R}^n$ action on φ_p by*

$$\omega : p \to \varphi_p^{\omega}(p) = \omega(\varphi_p(p)) = (\omega_1, \omega_2, \cdots, \omega_n),$$

i.e., if a line L passes through $\varphi(p)$ with direction angles $\theta_1, \theta_2, \cdots, \theta_n$ with axes $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ in $\mathbf{R}^n$, then its direction becomes

$$\theta_1 - \frac{\vartheta_1}{2} + \sigma_1, \theta_2 - \frac{\vartheta_2}{2} + \sigma_2, \cdots, \theta_n - \frac{\vartheta_n}{2} + \sigma_n$$

after passing through $\varphi_p(p)$, where for any integer $1 \le i \le n$, $\omega_i \equiv \vartheta_i(mod 4\pi)$, $\vartheta_i \ge 0$ and

$$\sigma_i = \begin{cases} \pi, & if \quad 0 \le \omega_i < 2\pi, \\ 0, & if \quad 2\pi < \omega_i < 4\pi. \end{cases}$$

A manifold M^n endowed with such a spatially directional mapping $\omega : M^n \to \mathbf{R}^n$ is called an n-dimensional pseudo-manifold, denoted by $(M^n, \mathcal{A}^{\omega})$.

Theorem 2.1 *For a point $p \in M^n$ with local chart (U_p, φ_p), $\varphi_p^{\omega} = \varphi_p$ if and only if $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod 2)$ for $1 \le i \le n$.*

Proof By definition, for any point $p \in M^n$, if $\varphi_p^{\omega}(p) = \varphi_p(p)$, then $\omega(\varphi_p(p)) = \varphi_p(p)$. According to Construction 2.1, this can only happens while $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod 2)$ for $1 \le i \le n$. □

Definition 2.1 *A spatially directional mapping $\omega : M^n \to \mathbf{R}^n$ is euclidean if for any point $p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod 2)$ for $1 \le i \le n$, otherwise, non-euclidean.*

Definition 2.2 *Let $\omega : M^n \to \mathbf{R}^n$ be a spatially directional mapping and $p \in (M^n, \mathcal{A}^{\omega})$, $\omega(p)(mod 4\pi) = (\omega_1, \omega_2, \cdots, \omega_n)$. Call a point p elliptic, euclidean or hyperbolic in direction $\mathbf{e}_i$, $1 \le i \le n$ if $o \le \omega_i < 2\pi$, $\omega_i = 2\pi$ or $2\pi < \omega_i < 4\pi$.*

Then we get a consequence by Theorem 2.1.

Corollary 2.1 *Let $(M^n, \mathcal{A}^{\omega})$ be a pseudo-manifold. Then $\varphi_p^{\omega} = \varphi_p$ if and only if every point in M^n is euclidean.*

Theorem 2.2 *Let $(M^n, \mathcal{A}^\omega)$ be an n-dimensional pseudo-manifold and $p \in M^n$. If there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in (U_p, φ_p), then $(M^n, \mathcal{A}^\omega)$ is a Smarandache n-manifold.*

Proof On the first, we introduce a conception for locally parallel lines in an n-manifold. Two lines C_1, C_2 are said *locally parallel* in a neighborhood (U_p, φ_p) of a point $p \in M^n$ if $\varphi_p(C_1)$ and $\varphi_p(C_2)$ are parallel straight lines in $\mathbf{R}^n$.

In $(M^n, \mathcal{A}^\omega)$, the axiom that *there are lines pass through a point locally parallel a given line* is Smarandachely denied since it behaves in at least two different ways, i.e., *one parallel, none parallel*, or *one parallel, infinite parallels*, or *none parallel, infinite parallels*.

If there are euclidean and non-euclidean points in (U_p, φ_p) simultaneously, not loss of generality, we assume that u is euclidean but v non-euclidean, $\omega(v)(mod 4\pi) = (\omega_1, \omega_2, \cdots, \omega_n)$ and $\omega_1 \neq 2\pi$. Now let L be a straight line parallel the axis $\mathbf{e}_1$ in $\mathbf{R}^n$. There is only one line C_u locally parallel to $\varphi_p^{-1}(L)$ passing through the point u since there is only one line $\varphi_p(C_q)$ parallel to L in $\mathbf{R}^n$ by these axioms for Euclid spaces. However, if $0 < \omega_1 < 2\pi$, then there are infinite many lines passing through u locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) since there are infinite many straight lines parallel L in $\mathbf{R}^n$, such as those shown in Fig.2.1(a) in where each straight line passing through the point $\overline{u} = \varphi_p(u)$ from the shade field is parallel to L.

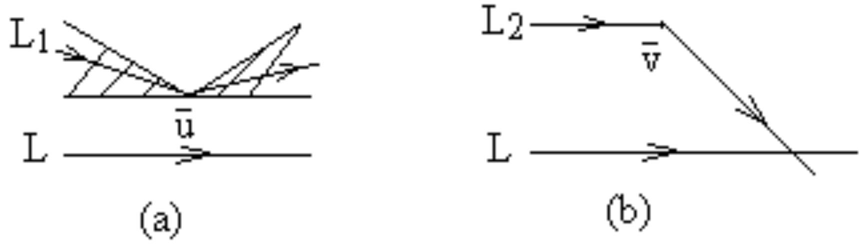

Fig.2.1

But if $2\pi < \omega_1 < 4\pi$, then there are no lines locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) since there are no straight lines passing through the point $\overline{v} = \varphi_p(v)$ parallel to L in $\mathbf{R}^n$, such as those shown in Fig.2.1(b).

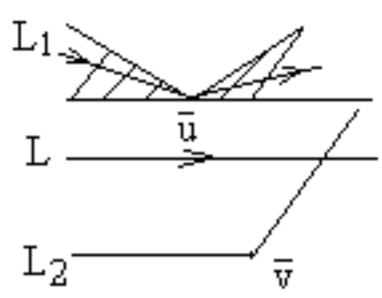

Fig.2.2

If there are two elliptic points u, v along a direction $\overrightarrow{O}$, consider the plane $\mathcal{P}$ determined

by $\omega(u), \omega(v)$ with $\overrightarrow{O}$ in $\mathbf{R}^n$. Let L be a straight line intersecting with the line uv in $\mathcal{P}$. Then there are infinite lines passing through u locally parallel to $\varphi_p(L)$ but none line passing through v locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) since there are infinite many lines or none lines passing through $\overline{u} = \omega(u)$ or $\overline{v} = \omega(v)$ parallel to L in $\mathbf{R}^n$, such as those shown in Fig.2.2.

Similarly, we can also get the conclusion for the case of hyperbolic points. Since there exists a Smarandachely denied axiom in $(M^n, \mathcal{A}^\omega)$, it is a Smarandache manifold. This completes the proof. $\square$

For an Euclid space $\mathbf{R}^n$, the homeomorphism φ_p is trivial for $\forall p \in \mathbf{R}^n$. In this case, we abbreviate $(\mathbf{R}^n, \mathcal{A}^\omega)$ to $(\mathbf{R}^n, \omega)$.

Corollary 2.2 *For any integer $n \geq 2$, if there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in $(\mathbf{R}^n, \omega)$, then $(\mathbf{R}^n, \omega)$ is an n-dimensional Smarandache geometry.*

Particularly, Corollary 2.2 partially answers an open problem in [12] for establishing Smarandache geometries in $\mathbf{R}^3$.

Corollary 2.3 *If there are points $p, q \in \mathbf{R}^3$ such that $\omega(p)(mod 4\pi) \neq (2\pi, 2\pi, 2\pi)$ but $\omega(q)(mod 4\pi) = (2\pi k_1, 2\pi k_2, 2\pi k_3)$, where $k_i \equiv 1(mod 2), 1 \leq i \leq 3$ or p, q are simultaneously elliptic or hyperbolic in a same direction of $\mathbf{R}^3$, then $(\mathbf{R}^3, \omega)$ is a Smarandache space geometry.*

Definition 2.3 *For any integer $r \geq 1$, a C^r differential Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$ is a Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$ endowed with a differential structure $\mathcal{A}$ and a C^r spatially directional mapping ω. A C^∞ Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$ is also said to be a smooth Smarandache n-manifold.*

According to Theorem 2.2, we get the next result by definitions.

Theorem 2.3 *Let $(M^n, \mathcal{A})$ be a manifold and $\omega : M^n \to \mathbf{R}^n$ a spatially directional mapping action on $\mathcal{A}$. Then $(M^n, \mathcal{A}^\omega)$ is a C^r differential Smarandache n-manifold for an integer $r \geq 1$ if the following conditions hold:*

(1) *there is a C^r differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n;*

(2) *ω is C^r;*

(3) *there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in (U_p, φ_p) for a point $p \in M^n$.*

Proof The condition (1) implies that $(M^n, \mathcal{A})$ is a C^r differential n-manifold and conditions (2), (3) ensure $(M^n, \mathcal{A}^\omega)$ is a differential Smarandache manifold by definitions and Theorem 2.2. $\square$

For a smooth differential Smarandache n-manifold $(M^n, \mathcal{A}^\omega)$, a function $f : M^n \to \mathbf{R}$ is said smooth if for $\forall p \in M^n$ with an chart (U_p, φ_p),

$$f \circ (\varphi_p^\omega)^{-1} : (\varphi_p^\omega)(U_p) \to \mathbf{R}^n$$

is smooth. Denote by $\Im_p$ all these C^∞ functions at a point $p \in M^n$.

Definition 2.4 *Let $(M^n, \mathcal{A}^\omega)$ be a smooth differential Smarandache n-manifold and $p \in M^n$. A tangent vector v at p is a mapping $v : \Im_p \to \mathbf{R}$* with these following conditions hold.

(1) $\forall g, h \in \Im_p, \forall \lambda \in \mathbf{R},\ v(h + \lambda h) = v(g) + \lambda v(h)$;

(2) $\forall g, h \in \Im_p, v(gh) = v(g)h(p) + g(p)v(h)$.

Denote all tangent vectors at a point $p \in (M^n, \mathcal{A}^\omega)$ by T_pM^n and define addition+and scalar multiplication·for $\forall u, v \in T_pM^n, \lambda \in \mathbf{R}$ and $f \in \Im_p$ by

$$(u + v)(f) = u(f) + v(f), \quad (\lambda u)(f) = \lambda \cdot u(f).$$

Then it can be shown immediately that T_pM^n is a vector space under these two operations+and·.

Let $p \in (M^n, \mathcal{A}^\omega)$ and $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ be a smooth curve in $\mathbf{R}^n$ with $\gamma(0) = p$. In $(M^n, \mathcal{A}^\omega)$, there are four possible cases for tangent lines on γ at the point p, such as those shown in Fig.2.3, in where these bold lines represent tangent lines.

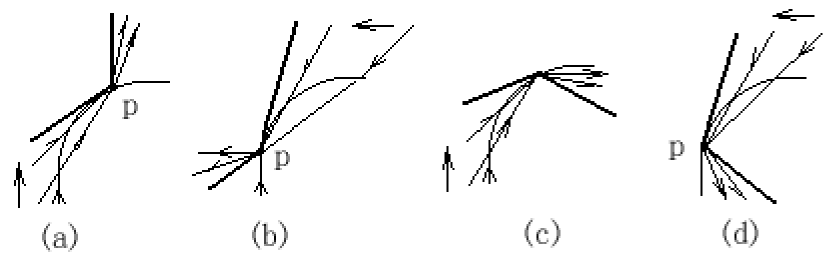

Fig.2.3

By these positions of tangent lines at a point p on γ, we conclude that *there is one tangent line at a point p on a smooth curve if and only if p is euclidean in $(M^n, \mathcal{A}^\omega)$*. This result enables us to get the dimensional number of a tangent vector space T_pM^n at a point $p \in (M^n, \mathcal{A}^\omega)$.

Theorem 2.4 *For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart (U_p, φ_p), $\varphi_p(p) = (x_1^0, x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of T_pM^n is*

$$\dim T_pM^n \ = \ 2n - s$$

with a basis

$$\{\frac{\partial}{\partial x^{i_j}}|_p \mid 1 \leq j \leq s\} \bigcup \{\frac{\partial^-}{\partial x^l}|_p, \frac{\partial^+}{\partial x^l}|_p \mid 1 \leq l \leq n \ and \ l \neq i_j, 1 \leq j \leq s\}.$$

Proof We only need to prove that

$$\{\frac{\partial}{\partial x^{i_j}}|_p \mid 1 \leq j \leq s\} \bigcup \{\frac{\partial^-}{\partial x^l}, \frac{\partial^+}{\partial x^l}|_p \mid 1 \leq l \leq n \ and \ l \neq i_j, 1 \leq j \leq s\} \quad (2.1)$$

is a basis of T_pM^n. For $\forall f \in \Im_p$, since f is smooth, we know that

$$\begin{aligned} f(x) &= f(p) + \sum_{i=1}^{n}(x_i - x_i^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p) \\ &+ \sum_{i,j=1}^{n}(x_i - x_i^0)(x_j - x_j^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}\frac{\partial^{\epsilon_j} f}{\partial x_j} + R_{i,j,\cdots,k} \end{aligned}$$

for $\forall x = (x_1, x_2, \cdots, x_n) \in \varphi_p(U_p)$ by the Taylor formula in $\mathbf{R}^n$, where each term in $R_{i,j,\cdots,k}$ contains $(x_i - x_i^0)(x_j - x_j^0)\cdots(x_k - x_k^0)$, $\epsilon_l \in \{+,-\}$ for $1 \leq l \leq n$ but $l \neq i_j$ for $1 \leq j \leq s$ and ϵ_l should be deleted for $l = i_j, 1 \leq j \leq s$.

Now let $v \in T_pM^n$. By Definition 2.4(1), we get that

$$\begin{aligned} v(f(x)) &= v(f(p)) + v(\sum_{i=1}^{n}(x_i - x_i^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p)) \\ &+ v(\sum_{i,j=1}^{n}(x_i - x_i^0)(x_j - x_j^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}\frac{\partial^{\epsilon_j} f}{\partial x_j}) + v(R_{i,j,\cdots,k}). \end{aligned}$$

Application of the condition (2) in Definition 2.4 shows that

$$v(f(p)) = 0, \quad \sum_{i=1}^{n} v(x_i^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p) = 0,$$

$$v(\sum_{i,j=1}^{n}(x_i - x_i^0)(x_j - x_j^0)\frac{\partial^{\epsilon_i} f}{\partial x_i}\frac{\partial^{\epsilon_j} f}{\partial x_j}) = 0$$

and

$$v(R_{i,j,\cdots,k}) = 0.$$

Whence, we get that

$$v(f(x)) = \sum_{i=1}^{n} v(x_i)\frac{\partial^{\epsilon_i} f}{\partial x_i}(p) = \sum_{i=1}^{n} v(x_i)\frac{\partial^{\epsilon_i}}{\partial x_i}|_p(f). \quad (2.2)$$

The formula (2.2) shows that any tangent vector v in T_pM^n can be spanned by elements in (2.1).

All elements in (2.1) are linearly independent. Otherwise, if there are numbers $a^1, a^2, \cdots, a^s$, $a_1^+, a_1^-, a_2^+, a_2^-, \cdots, a_{n-s}^+, a_{n-s}^-$ such that

$$\sum_{j=1}^{s} a_{i_j}\frac{\partial}{\partial x_{i_j}} + \sum_{i \neq i_1, i_2, \cdots, i_s, 1 \leq i \leq n} a_i^{\epsilon_i}\frac{\partial^{\epsilon_i}}{\partial x_i}|_p = 0,$$

where $\epsilon_i \in \{+,-\}$, then we get that

$$a_{i_j} = (\sum_{j=1}^{s} a_{i_j}\frac{\partial}{\partial x_{i_j}} + \sum_{i \neq i_1, i_2, \cdots, i_s, 1 \leq i \leq n} a_i^{\epsilon_i}\frac{\partial^{\epsilon_i}}{\partial x_i})(x_{i_j}) = 0$$

for $1 \leq j \leq s$ and

$$a_i^{\epsilon_i} = (\sum_{j=1}^{s} a_{i_j} \frac{\partial}{\partial x_{i_j}} + \sum_{i \neq i_1, i_2, \cdots, i_s, 1 \leq i \leq n} a_i^{\epsilon_i} \frac{\partial^{\epsilon_i}}{\partial x_i})(x_i) = 0$$

for $i \neq i_1, i_2, \cdots, i_s, 1 \leq i \leq n$. Therefore, (2.1) is a basis of the tangent vector space T_pM^n at the point $p \in (M^n, \mathcal{A}^\omega)$. $\square$

Notice that $\dim T_pM^n = n$ in Theorem 2.4 if and only if all these directions are euclidean along $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. We get a consequence by Theorem 2.4.

Corollary 2.4([4]-[5]) *Let $(M^n, \mathcal{A})$ be a smooth manifold and $p \in M^n$. Then*

$$\dim T_pM^n = n$$

with a basis

$$\{\frac{\partial}{\partial x^i}|_p \mid 1 \leq i \leq n\}.$$

Definition 2.5 *For $\forall p \in (M^n, \mathcal{A}^\omega)$, the dual space $T_p^*M^n$ is called a co-tangent vector space at p.*

Definition 2.6 *For $f \in \Im_p, d \in T_p^*M^n$ and $v \in T_pM^n$, the action of d on f, called a differential operator $d : \Im_p \to \mathbf{R}$, is defined by*

$$df = v(f).$$

Then we immediately get the following result.

Theorem 2.5 *For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart (U_p, φ_p), $\varphi_p(p) = (x_1^{\cdot} x_2^0, \cdots, x_n^0)$, if there are just s euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of $T_p^*M^n$ is*

$$\dim T_p^*M^n = 2n - s$$

with a basis

$$\{dx^{i_j}|_p \mid 1 \leq j \leq s\} \bigcup \{d^- x_p^l, d^+ x^l|_p \mid 1 \leq l \leq n \text{ and } l \neq i_j, 1 \leq j \leq s\},$$

where

$$dx^i|_p(\frac{\partial}{\partial x^j}|_p) = \delta_j^i \text{ and } d^{\epsilon_i} x^i|_p(\frac{\partial^{\epsilon_i}}{\partial x^j}|_p) = \delta_j^i$$

for $\epsilon_i \in \{+, -\}, 1 \leq i \leq n$.

§3. Pseudo-Manifold Geometries

Here we introduce *Minkowski norms* on these pseudo-manifolds $(M^n, \mathcal{A}^\omega)$.

Definition 3.1 *A Minkowski norm on a vector space V is a function $F : V \to \mathbf{R}$ such that*

(1) *F is smooth on $V\backslash\{0\}$ and $F(v) \geq 0$ for $\forall v \in V$;*

(2) *F is 1-homogenous, i.e., $F(\lambda v) = \lambda F(v)$ for $\forall \lambda > 0$;*

(3) *for all $y \in V\backslash\{0\}$, the symmetric bilinear form $g_y : V \times V \to \mathbf{R}$ with*

$$g_y(u, v) = \sum_{i,j} \frac{\partial^2 F(y)}{\partial y^i \partial y^j}$$

is positive definite for $u, v \in V$.

Denote by $TM^n = \bigcup_{p\in(M^n,\mathcal{A}^\omega)} T_pM^n$.

Definition 3.2 *A pseudo-manifold geometry is a pseudo-manifold $(M^n, \mathcal{A}^\omega)$ endowed with a Minkowski norm F on TM^n.*

Then we get the following result.

Theorem 3.1 *There are pseudo-manifold geometries.*

Proof Consider an eucildean $2n$-dimensional space $\mathbf{R}^{2n}$. Then there exists a Minkowski norm $F(\overline{x}) = |\overline{x}|$ at least. According to Theorem 2.4, T_pM^n is $\mathbf{R}^{s+2(n-s)}$ if $\omega(p)$ has s euclidean directions along $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. Whence there are Minkowski norms on each chart of a point in $(M^n, \mathcal{A}^\omega)$.

Since $(M^n, \mathcal{A})$ has finite cover $\{(U_\alpha, \varphi_\alpha)|\alpha \in I\}$, where I is a finite index set, by the decomposition theorem for unit, we know that there are smooth functions $h_\alpha, \alpha \in I$ such that

$$\sum_{\alpha\in I} h_\alpha = 1 \text{ with } 0 \leq h_\alpha \leq 1.$$

Choose a Minkowski norm F^α on each chart $(U_\alpha, \varphi_\alpha)$. Define

$$F_\alpha = \begin{cases} h^\alpha F^\alpha, & \text{if} \quad p \in U_\alpha, \\ 0, & \text{if} \quad p \notin U_\alpha \end{cases}$$

for $\forall p \in (M^n, \varphi^\omega)$. Now let

$$F = \sum_{\alpha\in I} F_\alpha.$$

Then F is a Minkowski norm on TM^n since it satisfies all of these conditions $(1) - (3)$ in Definition 3.1. □

Although the dimension of each tangent vector space maybe different, we can also introduce *principal fiber bundles* and *connections* on pseudo-manifolds.

Definition 3.3 *A principal fiber bundle (PFB) consists of a pseudo-manifold $(P, \mathcal{A}_1^\omega)$, a projection $\pi : (P, \mathcal{A}_1^\omega) \to (M, \mathcal{A}_0^{\pi(\omega)})$, a base pseudo-manifold $(M, \mathcal{A}_0^{\pi(\omega)})$ and a Lie group G, denoted by (P, M, ω^π, G) such that (1), (2) and (3) following hold.*

(1) *There is a right freely action of* G *on* $(P, \mathcal{A}_1^\omega)$, *i.e., for* $\forall g \in G$, *there is a diffeomorphism* $R_g : (P, \mathcal{A}_1^\omega) \to (P, \mathcal{A}_1^\omega)$ *with* $R_g(p^\omega) = p^\omega g$ *for* $\forall p \in (P, \mathcal{A}_1^\omega)$ *such that* $p^\omega(g_1 g_2) = (p^\omega g_1)g_2$ *for* $\forall p \in (P, \mathcal{A}_1^\omega)$, $\forall g_1, g_2 \in G$ *and* $p^\omega e = p^\omega$ *for some* $p \in (P^n, \mathcal{A}_1^\omega)$, $e \in G$ *if and only if* e *is the identity element of* G.

(2) *The map* $\pi : (P, \mathcal{A}_1^\omega) \to (M, \mathcal{A}_0^{\pi(\omega)})$ *is onto with* $\pi^{-1}(\pi(p)) = \{pg | g \in G\}$, $\pi\omega_1 = \omega_0\pi$, *and regular on spatial directions of* p, *i.e., if the spatial directions of* p *are* $(\omega_1, \omega_2, \cdots, \omega_n)$, *then* ω_i *and* $\pi(\omega_i)$ *are both elliptic, or euclidean, or hyperbolic and* $|\pi^{-1}(\pi(\omega_i))|$ *is a constant number independent of* p *for any integer* $i, 1 \leq i \leq n$.

(3) *For* $\forall x \in (M, \mathcal{A}_0^{\pi(\omega)})$ *there is an open set* U *with* $x \in U$ *and a diffeomorphism* $T_u^{\pi(\omega)}$: $(\pi)^{-1}(U^{\pi(\omega)}) \to U^{\pi(\omega)} \times G$ *of the form* $T_u(p) = (\pi(p^\omega), s_u(p^\omega))$, *where* $s_u : \pi^{-1}(U^{\pi(\omega)}) \to G$ *has the property* $s_u(p^\omega g) = s_u(p^\omega)g$ *for* $\forall g \in G, p \in \pi^{-1}(U)$.

We know the following result for principal fiber bundles of pseudo-manifolds.

Theorem 3.2 *Let* (P, M, ω^π, G) *be a PFB. Then*

$$(P, M, \omega^\pi, G) = (P, M, \pi, G)$$

if and only if all points in pseudo-manifolds $(P, \mathcal{A}_1^\omega)$ *are euclidean.*

Proof For $\forall p \in (P, \mathcal{A}_1^\omega)$, let (U_p, φ_p) be a chart at p. Notice that $\omega^\pi = \pi$ if and only if $\varphi_p^\omega = \varphi_p$ for $\forall p \in (P, \mathcal{A}_1^\omega)$. According to Theorem 2.1, by definition this is equivalent to that all points in $(P, \mathcal{A}_1^\omega)$ are euclidean. □

Definition 3.4 *Let* (P, M, ω^π, G) *be a PFB with* $\dim G = r$. *A subspace family* $H = \{H_p | p \in (P, \mathcal{A}_1^\omega), \dim H_p = \dim T_{\pi(p)} M\}$ *of* TP *is called a connection if conditions (1) and (2) following hold.*

(1) *For* $\forall p \in (P, \mathcal{A}_1^\omega)$, *there is a decomposition*

$$T_p P = H_p \bigoplus V_p$$

and the restriction $\pi_*|_{H_p} : H_p \to T_{\pi(p)} M$ *is a linear isomorphism.*

(2) H *is invariant under the right action of* G, *i.e., for* $p \in (P, \mathcal{A}_1^\omega)$, $\forall g \in G$,

$$(R_g)_{*p}(H_p) = H_{pg}.$$

Similar to Theorem 3.2, the conception of connection introduced in Definition 3.4 is more general than the popular connection on principal fiber bundles.

Theorem 3.3(dimensional formula) *Let* (P, M, ω^π, G) *be a PFB with a connection* H. *For* $\forall p \in (P, \mathcal{A}_1^\omega)$, *if the number of euclidean directions of* p *is* $\lambda_P(p)$, *then*

$$\dim V_p = \frac{(\dim P - \dim M)(2\dim P - \lambda_P(p))}{\dim P}.$$

Proof Assume these euclidean directions of the point p being $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{\lambda_P(p)}$. By definition π is regular, we know that $\pi(\mathbf{e}_1), \pi(\mathbf{e}_2), \cdots, \pi(\mathbf{e}_{\lambda_P(p)})$ are also euclidean in $(M, \mathcal{A}_1^{\pi(\omega)})$. Now since

$$\pi^{-1}(\pi(\mathbf{e}_1)) = \pi^{-1}(\pi(\mathbf{e}_2)) = \cdots = \pi^{-1}(\pi(\mathbf{e}_{\lambda_P(p)})) = \mu = \text{ constant},$$

we get that $\lambda_P(p) = \mu\lambda_M$, where λ_M denotes the correspondent euclidean directions in $(M, \mathcal{A}_1^{\pi(\omega)})$. Similarly, consider all directions of the point p, we also get that $\dim P = \mu \dim M$. Thereafter

$$\lambda_M = \frac{\dim M}{\dim P}\lambda_P(p). \qquad (3.1)$$

Now by Definition 3.4, $T_pP = H_p \bigoplus V_p$, i.e.,

$$\dim T_pP = \dim H_p + \dim V_p. \quad (3.2)$$

Since $\pi_*|_{H_p} : H_p \to T_{\pi(p)}M$ is a linear isomorphism, we know that $\dim H_p = \dim T_{\pi(p)}M$. According to Theorem 2.4, we have formulae

$$\dim T_pP = 2\dim P - \lambda_P(p)$$

and

$$\dim T_{\pi(p)}M = 2\dim M - \lambda_M = 2\dim M - \frac{\dim M}{\dim P}\lambda_P(p).$$

Now replacing all these formulae into (3.2), we get that

$$2\dim P - \lambda_P(p) = 2\dim M - \frac{\dim M}{\dim P}\lambda_P(p) + \dim V_p.$$

That is,

$$\dim V_p = \frac{(\dim P - \dim M)(2\dim P - \lambda_P(p))}{\dim P}. \qquad \square$$

We immediately get the following consequence by Theorem 3.3.

Corollary 3.1 *Let (P, M, ω^π, G) be a PFB with a connection H. Then for $\forall p \in (P, \mathcal{A}_1^\omega)$,*

$$\dim V_p = \dim P - \dim M$$

if and only if the point p is euclidean.

Now we consider conclusions included in Smarandache geometries, particularly in pseudo-manifold geometries.

Theorem 3.4 *A pseudo-manifold geometry (M^n, φ^ω) with a Minkowski norm on TM^n is a Finsler geometry if and only if all points of (M^n, φ^ω) are euclidean.*

Proof According to Theorem 2.1, $\varphi_p^\omega = \varphi_p$ for $\forall p \in (M^n, \varphi^\omega)$ if and only if p is eucildean. Whence, by definition (M^n, φ^ω) is a Finsler geometry if and only if all points of (M^n, φ^ω) are euclidean. □

Corollary 3.1 *There are inclusions among Smarandache geometries, Finsler geometry, Riemann geometry and Weyl geometry following*

$$\begin{aligned}\{Smarandache\ geometries\} &\supset \{pseudo-manifold\ geometries\}\\ &\supset \{Finsler\ geometry\} \supset \{Riemann\ geometry\}\\ &\supset \{Weyl\ geometry\}.\end{aligned}$$

Proof The first and second inclusions are implied in Theorems 2.1 and 3.3. Other inclusions are known in a textbook, such as [4] – [5]. □

Now we consider complex manifolds. Let $z^i = x^i + \sqrt{-1}y^i$. In fact, any complex manifold M_c^n is equal to a smooth real manifold M^{2n} with a natural base $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ for $T_pM_c^n$ at each point $p \in M_c^n$. Define a *Hermite* manifold M_c^n to be a manifold M_c^n endowed with a Hermite inner product $h(p)$ on the tangent space $(T_pM_c^n, J)$ for $\forall p \in M_c^n$, where J is a mapping defined by

$$J(\frac{\partial}{\partial x^i}|_p) = \frac{\partial}{\partial y^i}|_p, \quad J(\frac{\partial}{\partial y^i}|_p) = -\frac{\partial}{\partial x^i}|_p$$

at each point $p \in M_c^n$ for any integer $i, 1 \leq i \leq n$. Now let

$$h(p) = g(p) + \sqrt{-1}\kappa(p), \quad p \in M_c^m.$$

Then a *Kähler manifold* is defined to be a Hermite manifold (M_c^n, h) with a closed κ satisfying

$$\kappa(X, Y) = g(X, JY), \ \forall X, Y \in T_pM_c^n, \forall p \in M_c^n.$$

Similar to Theorem 3.3 for real manifolds, we know the next result.

Theorem 3.5 *A pseudo-manifold geometry (M_c^n, φ^ω) with a Minkowski norm on TM^n is a Kähler geometry if and only if F is a Hermite inner product on M_c^n with all points of (M^n, φ^ω) being euclidean.*

Proof Notice that a complex manifold M_c^n is equal to a real manifold M^{2n}. Similar to the proof of Theorem 3.3, we get the claim. □

As a immediately consequence, we get the following inclusions in Smarandache geometries.

Corollary 3.2 *There are inclusions among Smarandache geometries, pseudo-manifold geometry and Kähler geometry following*

$$\begin{aligned}\{Smarandache\ geometries\} &\supset \{pseudo-manifold\ geometries\}\\ &\supset \{Kähler\ geometry\}.\end{aligned}$$

§4. Further Discussions

Undoubtedly, there are many and many open problems and research trends in pseudo-manifold geometries. Further research these new trends and solving these open problems will enrich one's knowledge in sciences.

Firstly, we need to get these counterpart in pseudo-manifold geometries for some important results in Finsler geometry or Riemann geometry.

4.1. **Stokes Theorem** *Let $(M^n, \mathcal{A})$ be a smoothly oriented manifold with the T_2 axiom hold. Then for $\forall \varpi \in A_0^{n-1}(M^n)$,*

$$\int_{M^n} d\varpi = \int_{\partial M^n} \varpi.$$

This is the well-known Stokes formula in Riemann geometry. If we replace $(M^n, \mathcal{A})$ by $(M^n, \mathcal{A}^\omega)$, *what will happens?* Answer this question needs to solve problems following.

(1) *Establish an integral theory on pseudo-manifolds.*

(2) *Find conditions such that the Stokes formula hold for pseudo-manifolds.*

4.2. **Gauss-Bonnet Theorem** *Let S be an orientable compact surface. Then*

$$\int\int_S K d\sigma = 2\pi\chi(S),$$

where K and $\chi(S)$ are the Gauss curvature and Euler characteristic of S This formula is the well-known Gauss-Bonnet formula in differential geometry on surfaces. Then *what is its counterpart in pseudo-manifold geometries?* This need us to solve problems following.

(1) *Find a suitable definition for curvatures in pseudo-manifold geometries.*

(2) *Find generalizations of the Gauss-Bonnect formula for pseudo-manifold geometries, particularly, for pseudo-surfaces.*

For a oriently compact Riemann manifold (M^{2p}, g), let

$$\Omega = \frac{(-1)^p}{2^{2p}\pi^p p!} \sum_{i_1, i_2, \cdots, i_{2p}} \delta_{1,\cdots,2p}^{i_1,\cdots,i_{2p}} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2p-1} i_{2p}},$$

where Ω_{ij} is the curvature form under the natural chart $\{e_i\}$ of M^{2p} and

$$\delta_{1,\cdots,2p}^{i_1,\cdots,i_{2p}} = \begin{cases} 1, & \text{if permutation } i_1 \cdots i_{2p} \text{ is even}, \\ -1, & \text{if permutation } i_1 \cdots i_{2p} \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Chern proved that[4]−[5]

$$\int_{M^{2p}} \Omega = \chi(M^{2p}).$$

Certainly, these new kind of global formulae for pseudo-manifold geometries are valuable to find.

4.3. **Gauge Fields** Physicists have established a gauge theory on principal fiber bundles of Riemannian manifolds, which can be used to unite gauge fields with gravitation. Similar consideration for pseudo-manifold geometries will induce new gauge theory, which enables us to asking problems following.

Establish a gauge theory on those of pseudo-manifold geometries with some additional conditions.

(1) *Find these conditions such that we can establish a gauge theory on a pseudo-manifold geometry.*

(2) *Find the Yang-Mills equation in a gauge theory on a pseudo-manifold geometry.*

(2) *Unify these gauge fields and gravitation.*

References

[1] R.Abraham, J.E.Marsden and T.Ratiu, *Manifolds, tensor analysis, and applications*, Addison-Wesley Publishing Company, Inc. 1983.

[2] G.Bassini and S.Capozziello, Multi-Spaces and many worlds from conservation laws, *Progress in Physics*, Vol.4(2006), 65-72.

[3] D.Bleecker, *Gauge theory and variational principles*, Addison-Wesley Publishing Company, Inc. 1981.

[4] S.S.Chern and W.H.Chern, *Lectures in Differential Geometry*(in Chinese), Peking University Press, 2001.

[5] W.H.Chern and X.X.Li, *Introduction to Riemannian Geometry*, Peking University Press, 2002.

[6] H.Iseri, *Smarandache manifolds*, American Research Press, Rehoboth, NM,2002.

[7] H.Iseri, *Partially Paradoxist Smarandache Geometries*, http://www.gallup.unm. edu/s̃marandache/Howard-Iseri-paper.htm.

[8] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *Mathematics Magazine, Aurora, Canada*, Vol.12(2003).

[9] L.F.Mao, *On Automorphisms groups of Maps, Surfaces and Smarandache geometries*, *Sientia Magna*, Vol.1(2005), No.2, 55-73.

[10] L.F.Mao, A new view of combinatorial maps by Smarandache's notion, in *Selected Papers on Mathematical Combinatorics*(I), World Academic Union, 2006, also in *arXiv: math.GM/0506232*.

[11] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[12] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.

[13] L.F.Mao, On multi-metric spaces, *Scientia Magna*, Vol.2, No.1(2006), 87-94.

[14] F.Smarandache, *A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[15] F.Smarandache, A Unifying Field in Logic: Neutrosophic Field, *Multi-Valued Logic*, Vol.8, No.3(2002)(special issue on Neutrosophy and Neutrosophic Logic), 385-438.

[16] F.Smarandache, Mixed non-euclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

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Minimum Cycle Base of Graphs Identified by Two Planar Graphs

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Abstract: In this paper, we study the minimum cycle base of the planar graphs obtained from two 2-connected planar graphs by identifying an edge (or a cycle) of one graph with the corresponding edge (or cycle) of another, related with map geometries, i.e., Smarandache 2-dimensional manifolds. Also, we give a formula for calculating the length of minimum cycle base of a planar graph $N(d, \lambda)$ defined in paper [11].

Key Words: graph, planar graph, cycle space, minimum cycle base.
AMS(2000) : O5C10

§1. Introduction

Throughout this paper we consider simple and undirected graphs. The cardinality of a set A is $|A|$. Let's begin with some terminologies and some facts about cycle bases of graphs. Let $G(V, E)$ be a 2-connected graph with vertex set V and edge set E. The set $\mathcal{E}$ of all subsets of E forms an $|E|$-dimensional vector space over $GF(2)$ with vector addition $X \oplus Y = (X \cup Y) \backslash (X \cap Y)$ and scalar multiplication $1 \bullet X = X, 0 \bullet X = \emptyset$ for all $X, Y \in \mathcal{E}$. A *cycle* is a connected graph whose any vertex degree is 2. The set $\mathcal{C}$ of all cycles of G forms a subspace of $(\mathcal{E}, \oplus, \bullet)$ which is called the *cycle space* of G. The dimension of the cycle space $\mathcal{C}$ is the Betti number of G, say $\beta(G)$, which is equal to $|E(G)| - |V(G)| + 1$. A base $\mathcal{B}$ of the cycle space of G is called a *cycle base* of G.

The length $|C|$ of a cycle C is the number of its edges. The length $l(\mathcal{B})$ of a cycle base $\mathcal{B}$ is the sum of lengths of all its cycles. A *minimum cycle base* (or MCB in short) is a cycle base with minimal length. A graph may has many minimum cycle bases, but every two minimum cycle bases have the same length.

Let G be a 2-connected planar graph embedded in the plane. G has $|E(G)| - |V(G)| + 2$ faces by Euler formula. There is exactly one face of G being unbounded which is called the

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exterior of G. All faces but the exterior of G are called interior faces of G. Each interior face of G has a cycle as its boundary which is called an *interior facial cycle*. Also, the cycle of G being incident with the exterior of G is called *the exterior facial cycle*.

We know that if G is a 2-connected planar graph embedded in the plane, then any set of $|E(G)| - |V(G)| + 1$ facial cycles forms a cycle base of G. For a 2-connected planar graph, we ask whether there is a minimum cycle base such that each cycle is a facial cycle. The answer isn't confirmed. The counterexample is easy to be constructed by Lemma 1.1. Need to say that Lemma 1.1 is a special case of Theorem A in the reference [10] which is deduced by Hall Theorem.

Lemma 1.1 *Let $\mathcal{B}$ be a cycle base of a 2-connected graph G. Then $\mathcal{B}$ is a minimum cycle base of G if and only if for any cycle C of G and cycle B in $\mathcal{B}$, if $B \in Int(C)$, then $|C| \geq |B|$, where Int (C) denotes the set of cycles in $\mathcal{B}$ which generate C.*

For some special 2-connected planar graph, there exist a minimum cycle base such that each cycle is a facial cycle. For example, Halin graph and outerplanar graph are such graphs. A *Halin graph* $H(T)$ consists of a tree T embedded in the plane without subdivision of an edge together with the additional edges joining the 1-valent vertices consecutively in their order in the planar embedding. It is clear that a Halin graphs is a 3-connected planar graph. The exterior facial cycle is called *leaf-cycle*.

Lemma 1.2[9,12] *Let $H(T)$ be a Halin graph embedded in the plane such that the leaf-cycle is the exterior facial cycle. Let $\mathcal{F}$ denote the set of interior facial cycles of $H(T)$. Then $\mathcal{F}$ is a minimum cycle base of $H(T)$.*

A planar graph G is outerplanar if it can be embedded in the plane such that all vertices lie on the exterior facial cycle C.

Lemma 1.3[6,9] *Let $G(V, E)$ be a 2-connected outerplanar graph embedded in the plane with C as its exterior facial cycle. Let $\mathcal{F}$ be the set of interior facial cycles. Then $\mathcal{F}$ is the minimum cycle base of G, and $l(\mathcal{F}) = 2|E| - |V|$.*

Apart from the above mentioned minimum cycle bases of a Halin graph and an outerplanar graph, many peoples researched minimum cycle bases of graphs. H. Ren et al. [9] not only gave a sufficient and necessary condition for minimum cycle base of a 2-connected planar graph, but also studied minimum cycle bases of graphs embedded in non-spherical surfaces and presented formulae for length of minimum cycle bases of some graphs such as the generalized Petersen graphs, the circulant graphs, etc. W.Imrich et al. [4] studied the minimum cycle bases for the cartesian and strong product of two graphs. P.Vismara [13] discussed the union of all the minimum cycle bases of a graph. What about the minimum cycle base of the graph obtained from two 2-connected planar graphs by identifying some corresponding edges? This problem is related with *map geometries*, i.e., *Smarandache 2-dimensional manifolds* (see [8] for details). We will consider it in this paper.

§2. MCB of graphs obtained by identifying an edge of planar graphs

Let G_1 and G_2 be two graphs and P_i be a path (or a cycle) in G_i for $i = 1, 2$. Suppose the length of P_1 is same as that of P_2. By identifying P_1 with P_2, we mean that the vertices of P_1 are identified with the corresponding vertices of P_2 and the multiedges are deleted.

Theorem 2.1 *Let G_1 and G_2 be two 2-connected planar graphs embedded in the plane. Let e_i be an edge in $E(G_i)$ such that e_i is in the exterior facial cycle of G_i for $i = 1, 2$. Let G be the graph obtained from G_1 and G_2 by identifying e_1 and e_2 such that G_2 is in the exterior of G_1. If the set of interior facial cycles of G_i, say $\mathcal{F}_i$, is a minimum cycle base of G_i for $i = 1, 2$, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a minimum cycle base of G.*

Proof Obviously, the graph G is a 2-connected planar graph and each cycle of $\mathcal{F}_1 \cup \mathcal{F}_2$ is a facial cycle of G. Since $|E(G)| = |E(G_1)| + |E(G_2)| - 1$ and $|V(G)| = |V(G_1)| + |V(G_2)| - 2$, G has $|E(G)| - |V(G)| + 2 = (|E(G_1)| - |V(G_1)| + 1) + (|E(G_2)| - |V(G_2)| + 1) + 1 = |\mathcal{F}_1| + |\mathcal{F}_2| + 1$ faces. So $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| = |E(G)| - |V(G)| + 1$, and $\mathcal{F}$ is a cycle base of G.

Now we prove that $\mathcal{F}$ is a minimum cycle base of G. Suppose F is a cycle of G and $F = f_1 \oplus f_2 \oplus \cdots \oplus f_q$, where $f_j \in \mathcal{F}$ for $j = 1, 2, \cdots, q$. By Lemma 1.1, We need to prove $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$.

If $E(F) \subset E(G_1)$ (or $E(G_2)$), then f_j is in $\mathcal{F}_1$ (or $\mathcal{F}_2$) for $j = 1, 2, \cdots, q$. By the fact that $\mathcal{F}_i$ is a minimum cycle base of G_i for $i = 1, 2$ and Lemma 1.1, $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$.

Let e be the edge of G obtained by e_1 identified with e_2. Suppose $e = \{uv\}$. If edges of F aren't in G_1 entirely, then F must pass through u and v. So $e \cup F$ can be partitioned into two cycles, say F_1 and F_2. Suppose $E(F_i) \subset E(G_i)$ for $i = 1, 2$. Then $|F| > |F_i|$ for $i = 1, 2$. Suppose $F_1 = f_1 \oplus f_2 \oplus \cdots \oplus f_p$ and $F_2 = f_{p+1} \oplus f_{p+2} \oplus \cdots \oplus f_q$. By the fact that $\mathcal{F}_i$ is a minimum cycle base of G_i for $i = 1, 2$ and Lemma 1.1, $|F| > |F_1| \geq |f_i|$ for $i = 1, 2, \cdots, p$ and $|F| > |F_2| \geq |f_i|$ for $i = p+1, p+2, \cdots, q$.

Thus we complete the proof. □

Applying Theorem 2.1 and the induction principle, it is easy to prove the following conclusion.

Corollary 2.1 *Let $G_1, G_2, \cdots, G_k$ be $k (k \geq 3)$ 2-connected planar graphs embedded in the plane. Let e_i be an edge in $E(G_i)$ such that e_i is in the exterior facial cycle of G_i for $i = 1, 2, \cdots, k$. Let G'_1 be the graph obtained from G_1 and G_2 by identifying e_1 with e_2 such that G_2 is in the exterior of G_1, Let G'_2 be the graph obtained from G'_1 and G_3 by identifying e_3 with some edge in the exterior face of G'_1 such that G_3 is in the exterior of G'_1, and so on. Let G be the last obtained graph in the above process. If the set of interior facial cycles of G_i, say $\mathcal{F}_i$, is a minimum cycle base of G_i for $i = 1, 2, \cdots, k$, then $\cup_{i=1}^{k} \mathcal{F}_i$ is a minimum cycle base of G.*

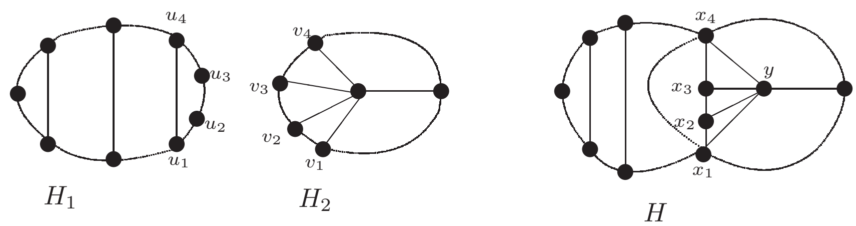

Fig.2.1

Remark: In Theorem 2.1, if e_1 is replaced by a path with length at least two and e_2 by the corresponding path, then the conclusion of the theorem doesn't hold. We consider the graph H shown in Fig.2.1, where H is obtained from H_1 and H_2 by identified $P_1 = u_1u_2u_3u_4$ with $P_2 = v_1v_2v_3v_4$. For the graph H, let $C = x_1x_2x_3x_4x_1$ and $D = x_1yx_4x_1$. Since $|C| > |D|$, the set of interior facial cycle of H isn't its minimum cycle base by Lemma 1.1.

Furthermore, if e_1 is replaced by a cycle and e_2 by the corresponding cycle in Theorem 2.1, then the conclusion of Theorem isn't true. The counterexample is easy to construct, which is left to readers. But if G_1 is a special planar graph, similar results to Theorem 2.1 will be shown in the next section.

§3. MCB of graphs obtained by identifying a cycle of planar graphs

An $r \times s$ cylinder is the graph with r radial lines and s cycles, where $r \geq 0, s > 0$. A 4×3 cylinder is shown in Fig.3.1. The innermost cycle is called the *central cycle*. $r \times s$ cylinder take an important role in discussion of the minor of planar graph with sufficiently large tree-width in paper[10].

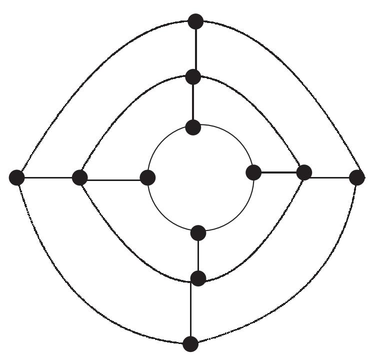

Fig.3.1

Theorem 3.1 *Let G_1 be an $r \times s(r \geq 4)$ cylinder embedded in the plane such that C is its central cycle. Let G_2 be a planar graph embedded in the plane such that the exterior facial cycle D has the same vertices as that of C. Let G be the graph obtained from G_1 and G_2 by identifying C and D such that G_2 is in the interior of G_1. If the set of interior facial cycles of G_2, say $\mathcal{F}_2$, is its a minimum cycle base, then the set of interior facial cycles of G, say $\mathcal{F}$, is a minimum cycle base of G.*

Proof At first, $\mathcal{F}$ is a cycle base of G. We need prove $\mathcal{F}$ is minimal.

Let $\mathcal{F}_1 = \mathcal{F} \backslash \mathcal{F}_2$. Obviously, each element of $\mathcal{F}_1$ has length 4. Suppose F is a cycle of G and $F = f_1 \oplus f_2 \oplus \cdots \oplus f_q$, where $f_j \in \mathcal{F}$ for $j = 1, 2, \cdots, q$. If we prove $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$, then $\mathcal{F}$ is a minimum cycle base of G by Lemma 1.1.

Let R be the open region bounded by F, and R' be the open region bounded by C (or D) of G_1 (or G_2). We consider the following four cases.

Case 1 $R' \cap R = \emptyset$.

Then F is a cycle of G_1 and F is generated by $\mathcal{F}_1$. Since the girth of G_1 is 4, $|F| \geq |f_j| = 4$ for $j = 1, 2, \cdots, q$.

Case 2 $R' \subset R$.

Then $|F| \geq |C| \geq 4$, because the number of radial lines which F crosses can't be less than the number of vertices of C. For a fixed f_j, if it is in the interior of C then $|f_j| \leq |C| \leq |F|$ by Lemma 1.1, because $\mathcal{F}_2$ is a minimum cycle base of G_2. If f_j is in the exterior of C, then $|f_j| = 4$. So $|f_i| \leq |F|$ for $j = 1, 2, \cdots, q$.

Case 3 $R \subset R'$.

Then F is a cycle of G_2. By Lemma 1.1, $|F| \geq |f_j|$ for $j = 1, 2, \cdots, q$.

Case 4 $R' \cap R \neq \emptyset$ and R' is not in the interior of R.

Then F must has at least one edge in $E(G_2) \backslash E(C)$ and at least three edges in $E(G_1)$. So $|F| \geq 4$. No loss of generality, suppose $f_1, f_2, \cdots, f_p$ are cycles of $\{f_1, f_2, \cdots, f_q\}$ that are in the exterior of C. Since $|f_j| = 4$, $|F| \geq |f_j|$ for $j = 1, 2, \cdots, p$.

Next we prove $|F| \geq |f_j|$ for $j = p+1, p+2, \cdots, q$, where f_j is in the interior of C.

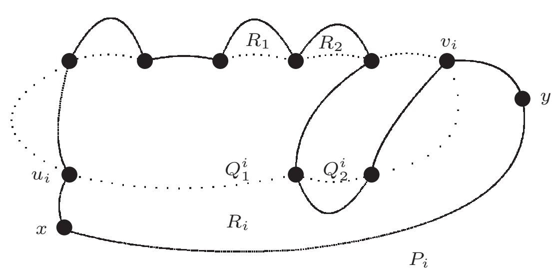

Fig. 3.2

Let $R" = R \backslash (R' \cap R)$. $R"$ may be the union of several regions. Let $R" = R_1 \cup R_2 \cup \cdots \cup R_l$ satisfying the condition that $R_i \cap R_j$ is empty or a point for $i \neq j, 1 \leq i, j \leq l$. Let B_i be the boundary of R_i for $i = 1, 2, \cdots, l$. Then B_i is a cycle in the exterior of C. For a fixed B_i, there may be many vertices of B_i in $V(F) \cap V(C)$, which can be found in Fig.3.2. We select two vertices u_i and v_i of B_i satisfying the following conditions:

(1) u_i and v_i are in C;

(2) there is a path of B_i, say P_i, such that its endvertices are u_i and v_i and P_i is in the exterior of C;

(3) if M_i is the path of B_i deleted $E(P_i)$, and if M_i' is the path of C such that its endvertices are u_i and v_i and M_i' is internally disjoint from B_i, then M_i is in the interior of the cycle which is the union of M_i' and P_i.

Note that M_i may contains many disjoint paths of C, suppose they are $Q_1^i, Q_2^i, \cdots, Q_t^i$. Let x, y be two vertices in P_i, which are adjacent to u_i, v_i respectively.

Obviously, x, y are in G_1. Let P_i' be the subpath of P_i between x and y. Considering the number of radial lines (including radial line x, y lie on) which P_i' crosses is not less than the number of vertices of $\cup_{j=1}^t Q_j^i$, $|P_i| > |P_i'| \geq \sum_{j=1}^t |Q_j^i|$.

Since $R' \cap R$ may be the union of some regions, we suppose $R' \cap R = D_1 \cup D_2 \cup \cdots \cup D_s$. Let $A_1, A_2, \cdots, A_s$ be boundaries of $D_1, D_2, \cdots, D_s$ respectively. For a fixed A_i, its edges may be partitioned into two groups, one containing edges of F, denoted as A_i^F, another containing edges of C, denoted as A_i^C. Then

$$\begin{aligned}\sum\nolimits_{i=1}^s |A_i| &= \sum\nolimits_{i=1}^s |A_i^F| + \sum\nolimits_{i=1}^s |A_i^C| \\ &= \sum\nolimits_{i=1}^s |A_i^F| + \sum\nolimits_{i=1}^l \sum\nolimits_{j=1}^t |Q_j^i| \\ &< \sum\nolimits_{i=1}^s |A_i^F| + \sum\nolimits_{i=1}^s |P_i| \\ &< |F|\end{aligned}$$

Hence $|F| > |A_i|$ for $i = 1, 2, \cdots, s$. Since any A_i is a cycle of G_2 and $\mathcal{F}_1$ is a minimum cycle base of G_2, $|A_i| \geq |f_j|$ for $j = i_1, i_2, \cdots, i_n$, by lemma 2.1, where $\{i_1, i_2, \cdots, i_n\} \subset \{p+1, p+2, \cdots, q\}$. Hence, $|F| > |f_{p+j}|$ for $i = 1, 2, \cdots, q-p$.

By the previous discussion and Lemma 1.1, $\mathcal{F}$ is a minimum cycle base of G. □

Since the minimum cycle base of a cycle is itself, a minimum cycle base of an $r \times s (r \geq 4)$ cylinder embedded in the plane is the set of its interior facial cycles by Theorem 3.1, and the length of its MCB is $r + 4r(s-1) = r(4s-3)$.

By Lemmas 1.2, 1.3 and Theorem 3.1. we get two corollaries following.

Corollary 3.1 *Assume an $r \times s (r \geq 4)$ cylinder, a Halin graph $H(T)$ are embedded in the plane with C the central cycle and C' the leaf-cycle of $H(T)$ containing the same vertices as C, respectively. Let G be the graph obtained from the $r \times s$ cylinder and $H(T)$ by identifying C and C' such that $H(T)$ is in the interior of the $r \times s$ cylinder. Then a minimum cycle base of G is the set of interior facial cycles of G.*

Corollary 3.2 *Assume an $r \times s (r \geq 4)$ cylinder, a 2-connected outplanar graph H be embedded in the plane with C the central cycle and C' the exterior facial cycle containing same vertices as C of H containing the same vertices as C, respectively. Let G be the graph obtained from the $r \times s$ cylinder and H by identifying C and C' such that H is in the interior of the $r \times s$ cylinder. Then a minimum cycle base of G is the set of interior facial cycles of G. Furthermore, the length of a MCB of G is $r(4s-5) + 2|E(H)|$.*

Proof Let $\mathcal{F}$ be the set of interior facial cycles of G. By Theorem 3.1, $\mathcal{F}$ is a minimum cycle base of G. $\mathcal{F}$ can be partitioned into two groups $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_1$ is the set of interior facial cycles of H and $\mathcal{F}_2$ the set of 4-cycles. Then the length of a MCB of G is $l(\mathcal{F}) = l(\mathcal{F}_1) + l(\mathcal{F}_2) = 4r(s-1) + 2|E(H)| - |V(H)| = (4s-5)r + 2|E(H)|$. $\square$

As application of Corollary 3.1, we find a formula for the length of minimum cycle base of a planar graph $N(d,\lambda)$, which can be found in paper[10].

When $\lambda \geq 1$ is an integer, the graph Y_λ is tree as shown in Fig.3.3. Thus Y_λ has $3 \times 2^{\lambda-1}$ 1-valent vertices and Y_λ has $3 \times 2^\lambda - 2$ vertices. If 1-valent vertices of Y_λ are connected in their order in the planar embedding, we obtain a special Halin graph, denoted by $H(\lambda)$.

Suppose a $(3 \times 2^{\lambda-1}) \times d$ cylinder is embedded in the plane such that its central cycle C has $3 \times 2^{\lambda-1}$ vertices. The graph obtained from $(3 \times 2^{\lambda-1}) \times d$ cylinder and $H(\lambda)$ with leaf-cycle C' containing $3 \times 2^{\lambda-1}$ vertices by identifying C and C' such that $H(\lambda)$ is in the interior of $(3 \times 2^{\lambda-1}) \times d$ cylinder is denoted as $N(d,\lambda)$. N. Roberterson and P.D. Seymour[10] proved that for all $d \geq 1, \lambda \geq 1$ the graph $N(d,\lambda)$ has tree-width $\leq 3d+1$.

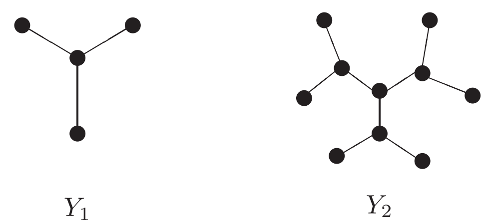

Fig.3.3

Theorem 3.2 *The length of minimum cycle base of $N(d,\lambda)$ ($\lambda \geq 2$) is* $3(d-1) \times 2^{\lambda+1} + 9 \times 2^\lambda - 3 \times 2^{\lambda-1} - 6$.

Proof Let $\mathcal{F}$ be the set of interior facial cycles of $N(d,\lambda)$. Then $\mathcal{F}$ is a minimum cycle base of $N(d,\lambda)$ by Corollary 3.1.

Let $\mathcal{F}_1$ be a subset of $\mathcal{F}$ which is the set of interior facial cycles of $N(1,\lambda)$ (a Halin graph). Then $\mathcal{F}_1$ consists of 3 $(2\lambda+1)$-cycles and 3×2^j $(2\lambda-2j-1)$-cycles for $j = 0, 1, 2, \cdots, \lambda-2$.

Let $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Then each cycle of $\mathcal{F}_2$ has length 4. Since the leaf-cycle of $N(1,\lambda)$ has $3 \times 2^{\lambda-1}$ vertices, there are $3(d-1) \times 2^{\lambda-1}$ 4-cycles in $\mathcal{F}_2$ all together. The length of $\mathcal{F}$ is

$$\begin{aligned}
l(\mathcal{F}) &= \sum_{j=0}^{\lambda-2} 3 \times 2^{j-1}(2\lambda-2j-1) + 3(2\lambda+1) + 4 \times 3(d-1) \times 2^{\lambda-1} \\
&= 3[\sum_{j=0}^{\lambda-2} \lambda 2^{j+1} - 2\sum_{j=0}^{\lambda-2} j2^j - \sum_{j=0}^{\lambda-2} 2^j] + (6\lambda+3) + 3(d-1) \times 2^{\lambda+1} \\
&= 3[(\lambda 2^\lambda - 2\lambda) - 2(\lambda-3)2^{\lambda-1} - 4 - 2^{\lambda-1} + 1] \\
&+ (6\lambda+3) + 3(d-1) \times 2^{\lambda+1} \\
&= 3(d-1) \times 2^{\lambda+1} + 9 \times 2^\lambda - 3 \times 2^{\lambda-1} - 6
\end{aligned}$$

Hence, the length of minimum cycle base of $N(d,\lambda)$ is $3(d-1) \times 2^{\lambda+1} + 9 \times 2^\lambda - 3 \times 2^{\lambda-1} - 6$. $\square$

Reference

[1] Bondy J.A. and Murty U.S.R., *Graph Theory with Application*, Macmillan, London, Elsevier, New York, 1976.

[2] Downs G.M., Gillet V.J., Holliday J.D. and Lynch M.F., Review of ring perception algorithms for chemical graph, *J. Chem.Inf.Comput.Sci.*,29: 172-187,1989.

[3] Horton J.D., A polynomial-time algorithm to find the shortest cycle base of a graph, *SIAM.J.Comput.*, 1987, 16: 359-366.

[4] Imrich W. and Stadler P.F., Minimum Cycle Bases of Product Graphs, *Australasian J. Comb.*, 26:233-244(2002).

[5] Mohar B. and Thomassen C., *Graphs on Surfaces*, Johns Hopkins University Press, 2001.

[6] Leydold J. and Stadler P.F., Minimum cycle bases of outerplanar graphs, *Electronic J. Combin.*, 5 # R16 (1998):1-14.

[7] Liu G.Z., On the connectivities of tree graphs, *J. Graph Theory*, 1988, 12: 453-459.

[8] L.F.Mao, *On Automorphisms groups of Maps, Surfaces and Smarandache geometries*, *Sientia Magna*, Vol.1(2005), No.2, 55-73.

[9] Ren H., Liu Y.P., Ma D.J. and Lu J.J., Generating cycle spaces for graphs on surfaces with small genera, *J. European of Combin.*, 2004, 25: 1087-1105.

[10] Ren H. and Deng M., Short cycle structures for graphs on surfaces and an open problem of Mohar and Thomassen, *Science in China: Series A math.*, 2006 Vol.49 No.2:212-224.

[11] Robertson N. and Seymour P.D., Graph minor. III. Planar tree-width, *J. Combinatorial Theory*, Series B 36 (1984): 49-64.

[12] Stadler P.F., Minimum cycle bases of Halin graphs, *J. Graph Theory*, 2003, 43:150-155

[13] Vismara P., Union of all the minimum cycle bases of a graph, *Electronic J. Combin.*, 4 #R9 (1997):1-15.

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A Combinatorially Generalized Stokes Theorem on Integrations

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Abstract: As an immediately application of Smarandache multi-spaces, a combinatorial manifold $\widetilde{M}$ with a given integer $m \geq 1$ is defined to be a geometrical object $\widetilde{M}$ such that for $\forall p \in \widetilde{M}$, there is a local chart (U_p, φ_p) enable $\varphi_p : U_p \to B^{n_{i_1}} \bigcup B^{n_{i_2}} \bigcup \cdots \bigcup B^{n_{i_{s(p)}}}$ with $B^{n_{i_1}} \bigcap B^{n_{i_2}} \bigcap \cdots \bigcap B^{n_{i_{s(p)}}} \neq \emptyset$, where $B^{n_{i_j}}$ is an n_{i_j}-ball for integers $1 \leq j \leq s(p) \leq m$. Integral theory on these smoothly combinatorial manifolds are introduced. Some classical results, such as those of *Stokes'* theorem and *Gauss'* theorem are generalized to smoothly combinatorial manifolds. By a relation of smoothly combinatorial manifolds with vertex-edge labeled graphs, counterparts of these conception and results are also established on graphs in this paper.

Key Words: combinatorial manifold, integration, Stokes' theorem, Gauss' theorem, vertex-edge labeled graph.

AMS(2000): 51M15, 53B15, 53B40, 57N16

§1. Introduction

As a localized Euclidean space, an *n-manifold* M^n is a Hausdorff space M^n, i.e., a space that satisfies the T_2 separation axiom such that for $\forall p \in M^n$, there is an open neighborhood $U_p, p \in U_p \subset M^n$ and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$. These manifolds, particularly, differential manifolds are very important to modern geometries and mechanics. As an immediately application of Smarandache multi-spaces ([8]), also the application of the combinatorial speculation for classical mathematics, i.e. *mathematics can be reconstructed from or turned into combinatorialization*([3]), combinatorial manifolds were introduced in [4], which are the generalization of classical manifolds and can be also endowed with a topological or differential structure as geometrical objects.

Now for an integer $s \geq 1$, let $n_1, n_2, \cdots, n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose s open unit balls $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$, where $\bigcap_{i=1}^{s} B_i^{n_i} \neq \emptyset$ in $\mathbf{R}^{n_1+n_2+\cdots n_s}$. A *unit open combinatorial ball of degree s* is a union

$$\widetilde{B}(n_1, n_2, \cdots, n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

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Then a combinatorial manifold $\widetilde{M}$ is defined in the next.

Definition 1.1 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ with $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ and $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension $\widehat{s}(p)$ of $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ are called the dimension and the intersectional dimensional of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ at the point p, denoted by $d(p)$ and $\widehat{d}(p)$, respectively.

A combinatorial manifold $\widetilde{M}$ is called *finite* if it is just combined by finite manifolds without one manifold is contained in the union of others, is called *smooth* if it is finite endowed with a C^∞ differential structure. For a smoothly combinatorial manifold $\widetilde{M}$ and a point $p \in \widetilde{M}$, it has been shown in [4] that $\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ and $\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ with a basis

$$\{\frac{\partial}{\partial x^{hj}}|_p | 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{\frac{\partial}{\partial x^{ij}}|_p \mid 1 \leq j \leq s\})$$

or

$$\{dx^{hj}|_p|\} 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{dx^{ij}|_p \mid 1 \leq j \leq s\}$$

for a given integer $h, 1 \leq h \leq s(p)$. Denoted all k-forms of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ by $\Lambda^k(\widetilde{M})$ and $\Lambda(\widetilde{M}) = \bigoplus\limits_{k=0}^{\widehat{s}(p)+\sum_{i=1}^{s(p)}(n_i - \widehat{s}(p))} \Lambda^k(\widetilde{M})$, then there is a unique exterior differentiation $\widetilde{d} : \Lambda(\widetilde{M}) \to \Lambda(\widetilde{M})$ such that for any integer $k \geq 1$, $\widetilde{d}(\Lambda^k) \subset \Lambda^{k+1}(\widetilde{M})$ with conditions following hold similar to the classical tensor analysis([1]).

(i) $\widetilde{d}$ is linear, i.e., for $\forall \varphi, \psi \in \Lambda(\widetilde{M})$, $\lambda \in \mathbf{R}$,

$$\widetilde{d}(\varphi + \lambda\psi) = \widetilde{d}\varphi \wedge \psi + \lambda \widetilde{d}\psi$$

and for $\varphi \in \Lambda^k(\widetilde{M}), \psi \in \Lambda(\widetilde{M})$,

$$\widetilde{d}(\varphi \wedge \psi) = \widetilde{d}\varphi + (-1)^k \varphi \wedge \widetilde{d}\psi.$$

(ii) For $f \in \Lambda^0(\widetilde{M})$, $\widetilde{d}f$ is the differentiation of f.

(iii) $\widetilde{d}^2 = \widetilde{d} \cdot \widetilde{d} = 0$.

(iv) $\widetilde{d}$ is a local operator, i.e., if $U \subset V \subset \widetilde{M}$ are open sets and $\alpha \in \Lambda^k(V)$, then $\widetilde{d}(\alpha|_U) = (\widetilde{d}\alpha)|_U$.

Therefore, smoothly combinatorial manifolds poss a local structure analogous smoothly manifolds. But notes that this local structure maybe different for neighborhoods of different points. Whence, geometries on combinatorial manifolds are *Smarandache* geometries([6]-[8]).

There are two well-known theorems in classical tensor analysis, i.e., *Stokes'* and *Gauss'* theorems for the integration of differential n-forms on an n-manifold M, which enables us knowing that

$$\int_M d\omega = \int_{\partial M} \omega$$

for a $\omega \in \Lambda^{n-1}(M)$ with compact supports and

$$\int_M (\mathrm{div} X)\mu = \int_{\partial M} \mathbf{i}_X \mu$$

for a vector field X, where $\mathbf{i}_X : \Lambda^{k+1}(M) \to \Lambda^k(M)$ defined by $\mathbf{i}_X \varpi(X_1, X_2, \cdots, X_k) = \varpi(X, X_1, \cdots, X_k)$ for $\varpi \in \Lambda^{k+1}(M)$. The similar local properties for combinatorial manifolds with manifolds naturally forward the following questions: *wether the Stokes' or Gauss' theorem is still valid on smoothly combinatorial manifolds?* or if invalid, *What are their modified forms for smoothly combinatorial manifolds?*.

The main purpose of this paper is to find the revised Stokes' or Gauss' theorem for combinatorial manifolds, namely, the Stokes' or Gauss' theorem is still valid for $\widetilde{n}$-forms on smoothly combinatorial manifolds $\widetilde{M}$ if $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n, m)$, where $\mathscr{H}_{\widetilde{M}}(n, m)$ is an integer set determined by its structure of a given smoothly combinatorial manifold $\widetilde{M}$. For this objective, we first consider a particular case of combinatorial manifolds, i.e., the combinatorial Euclidean spaces in the next section, establish a relation for finitely combinatorial manifolds with vertex-edge labeled graphs and calculate the integer set $\mathscr{H}_{\widetilde{M}}(n, m)$ for a given vertex-edge labeled graph in Section 3, then generalize the definition of integration on manifolds to combinatorial manifolds in Section 4. The generalized form for Stokes' or Gauss' theorem, also their counterparts on graphs can be found in Section 5. Terminologies and notations used in this paper are standard and can be found in $[1] - [2]$ or [4] for those of manifolds and combinatorial manifolds and [6] for graphs, respectively.

§2. Combinatorially Euclidean Spaces

As a simplest case of combinatorial manifolds, we characterize combinatorially Euclidean spaces of finite and generalize some results in Euclidean spaces in this section.

Definition 2.1 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a combinatorially Euclidean space $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is a union of finitely Euclidean spaces $\bigcup\limits_{i=1}^{m} \mathbf{R}^{n_i}$ such that for $\forall p \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, $p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ with $\widehat{m} = \dim(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i})$ a constant.*

By definition, we can present a point p of $\widetilde{\mathbf{R}}$ by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$.

$$[\overline{x}] = \left[\begin{array}{cccccccc} x^{11} & \cdots & x^{1\widehat{m}} & x^{1(\widehat{m})+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ x^{21} & \cdots & x^{2\widehat{m}} & x^{2(\widehat{m}+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x^{m1} & \cdots & x^{m\widehat{m}} & x^{m(\widehat{m}+1)} & \cdots & \cdots & x^{mn_m-1} & x^{mn_m} \end{array} \right]$$

For making a combinatorially Euclidean space to be a metric space, we introduce *inner product of matrixes* similar to that of vectors in the next.

Definition 2.2 *Let $(A) = (a_{ij})_{m\times n}$ and $(B) = (b_{ij})_{m\times n}$ be two matrixes. The inner product $\langle (A), (B)\rangle$ of (A) and (B) is defined by*

$$\langle (A), (B)\rangle = \sum_{i,j} a_{ij}b_{ij}.$$

Theorem 2.1 *Let $(A), (B), (C)$ be $m \times n$ matrixes and α a constant. Then*

(1) $\langle A, B\rangle = \langle B, A\rangle$;

(2) $\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$;

(3) $\langle \alpha A, B\rangle = \alpha \langle B, A\rangle$;

(4) $\langle A, A\rangle \geq 0$ with equality hold if and only if $(A) = O_{m\times n}$.

Proof (1)-(3) can be gotten immediately by definition. Now calculation shows that

$$\langle A, A\rangle = \sum_{i,j} a_{ij}^2 \geq 0$$

and with equality hold if and only if $a_{ij} = 0$ for any integers $i, j, 1 \leq i \leq m, 1 \leq j \leq n$, namely, $(A) = O_{m\times n}$. □

Theorem 2.2 *$(A), (B)$ be $m \times n$ matrixes. Then*

$$\langle (A), (B)\rangle^2 \leq \langle (A), (A)\rangle \langle (B), (B)\rangle$$

and with equality hold only if $(A) = \lambda(B)$, where λ is a real constant.

Proof If $(A) = \lambda(B)$, then $\langle A, B\rangle^2 = \lambda^2 \langle B, B\rangle^2 = \langle A, A\rangle \langle B, B\rangle$. Now if there are no constant λ enabling $(A) = \lambda(B)$, then $(A) - \lambda(B) \neq O_{m\times n}$ for any real number λ. According to Theorem 2.1, we know that

$$\langle (A) - \lambda(B), (A) - \lambda(B)\rangle > 0,$$

i.e.,

$$\langle (A), (A)\rangle - 2\lambda \langle (A), (B)\rangle + \lambda^2 \langle (B), (B)\rangle > 0.$$

Therefore, we find that

$$\Delta = (-2\langle (A),(B)\rangle)^2 - 4\langle (A),(A)\rangle\langle (B),(B)\rangle < 0,$$

namely,

$$\langle (A),(B)\rangle^2 < \langle (A),(A)\rangle\langle (B),(B)\rangle . \square$$

Corollary 2.1 *For given real numbers a_{ij}, b_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$,*

$$(\sum_{i,j} a_{ij}b_{ij})^2 \leq (\sum_{i,j} a_{ij}^2)(\sum_{i,j} b_{ij}^2).$$

Let O be the original point of $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$. Then $[O] = O_{m\times n_m}$. Now for $\forall p,q \in \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$, we also call $\overrightarrow{Op}$ the vector correspondent to the point p similar to that of classical Euclidean spaces, Then $\overrightarrow{pq} = \overrightarrow{Oq} - \overrightarrow{Op}$. Theorem 2.2 enables us to introduce an angle between two vectors $\overrightarrow{pq}$ and $\overrightarrow{uv}$ for points $p,q,u,v \in \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$.

Definition 2.3 *Let $p,q,u,v \in \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$. Then the angle θ between vectors $\overrightarrow{pq}$ and $\overrightarrow{uv}$ is determined by*

$$\cos\theta = \frac{\langle [p]-[q],[u]-[v]\rangle}{\sqrt{\langle [p]-[q],[p]-[q]\rangle\langle [u]-[v],[u]-[v]\rangle}}$$

under the condition that $0 \leq \theta \leq \pi$.

Corollary 2.2 *The conception of angle between two vectors is well defined.*

Proof Notice that

$$\langle [p]-[q],[u]-[v]\rangle^2 \leq \langle [p]-[q],[p]-[q]\rangle\langle [u]-[v],[u]-[v]\rangle$$

by Theorem 2.2. Thereby, we know that

$$-1 \leq \frac{\langle [p]-[q],[u]-[v]\rangle}{\sqrt{\langle [p]-[q],[p]-[q]\rangle\langle [u]-[v],[u]-[v]\rangle}} \leq 1.$$

Therefore there is a unique angle θ with $0 \leq \theta \leq \pi$ enabling Definition 2.3 hold. $\square$

For two points p,q in $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$, the *distance* $d(p,q)$ between points p and q is defined to be $\sqrt{\langle [p]-[q],[p]-[q]\rangle}$. We get the following result.

Theorem 2.3 *For a given integer sequence $n_1,n_2,\cdots,n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$ is a metric space.*

Proof We need to verify that each condition for a metric space holds in $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$. For two point $p,q \in \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$, by definition we know that

$$d(p,q) = \sqrt{\langle [p]-[q],[p]-[q]\rangle} \geq 0$$

with equality hold if and only if $[p]=[q]$, namely, $p=q$ and

$$d(p,q)=\sqrt{\langle [p]-[q],[p]-[q]\rangle}=\sqrt{\langle [q]-[p],[q]-[p]\rangle}=d(q,p).$$

Now let $u\in\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$. By Theorem 2.2, we then find that

$$\begin{aligned}
&(d(p,u)+d(u,p))^2\\
&=\langle [p]-[u],[p]-[u]\rangle+2\sqrt{\langle [p]-[u],[p]-[u]\rangle\langle [u]-[q],[u]-[q]\rangle}\\
&+\langle [u]-[q],[u]-[q]\rangle\\
&\geq \langle [p]-[u],[p]-[u]\rangle+2\langle [p]-[u],[u]-[q]\rangle+\langle [u]-[q],[u]-[q]\rangle\\
&=\langle [p]-[q],[p]-[q]\rangle=d^2(p,q).
\end{aligned}$$

Whence, $d(p,u)+d(u,p)\geq d(p,q)$ and $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$ is a metric space. □

By previous discussions, a combinatorially Euclidean space $\widetilde{R}(n_1,n_2,\cdots,n_m)$ can be turned to an Euclidean space $\mathbf{R}^n$ with $n=\widehat{m}+\sum\limits_{i=1}^{m}(n_i-\widehat{m})$. It is the same the other way round, namely we can also decompose an Euclidean space into a combinatorially Euclidean space.

Theorem 2.4 *Let $\mathbf{R}^n$ be an Euclidean space and $n_1,n_2,\cdots,n_m$ integers with $\widehat{m}<n_i<n$ for $1\leq i\leq m$ and the equation*

$$\widehat{m}+\sum_{i=1}^{m}(n_i-\widehat{m})=n$$

hold for an integer $\widehat{m}$, $1\leq\widehat{m}\leq n$. Then there is a combinatorially Euclidean space $\widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m)$ such that

$$\mathbf{R}^n\cong\widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m).$$

Proof Not loss of generality, assume the coordinate system of $\mathbf{R}^n$ is $(x_1,x_2,\cdots,x_n)$ with a basis $\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n\}$. Since

$$n-\widehat{m}=\sum_{i=1}^{m}(n_i-\widehat{m}),$$

Choose

$$\mathbf{R}_1=\langle \mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_{\widehat{m}},\mathbf{e}_{\widehat{m}+1},\cdots,\mathbf{e}_{n_1}\rangle;$$

$$\mathbf{R}_2=\langle \mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_{\widehat{m}},\mathbf{e}_{n_1+1},\mathbf{e}_{n_1+2},\cdots,\mathbf{e}_{n_2}\rangle;$$

$$\mathbf{R}_3=\langle \mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_{\widehat{m}},\mathbf{e}_{n_2+1},\mathbf{e}_{n_2+2},\cdots,\mathbf{e}_{n_3}\rangle;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$\mathbf{R}_m = \left\langle \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{\widehat{m}}, \mathbf{e}_{n_{m-1}+1}, \mathbf{e}_{n_{m-1}+2}, \cdots, \mathbf{e}_{n_m} \right\rangle.$$

Calculation shows $\dim\mathbf{R}_i = n_i$ and $\dim(\bigcap_{i=1}^{m} \mathbf{R}_i) = \widehat{m}$. Whence $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ is a combinatorially Euclidean space. By Definitions $2.1-2.2$ and Theorems $2.1-2.3$, we then get that

$$\mathbf{R}^n \cong \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m). \qquad \square$$

§3. Determining $\mathscr{H}_{\widetilde{M}}(n,m)$

Let $\widetilde{M}(n_1, \cdots, n_m)$ be a smoothly combinatorial manifold. Then there exists an atlas $\mathscr{C} = \{(\widetilde{U}_\alpha, [\varphi_\alpha]) | \alpha \in \widetilde{I}\}$ on $\widetilde{M}(n_1, \cdots, n_m)$ consisting of positively oriented charts such that for $\forall \alpha \in \widetilde{I}$, $\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ is an constant $n_{\widetilde{U}_\alpha}$ for $\forall p \in \widetilde{U}_\alpha$ ([4]). The integer set $\mathscr{H}_{\widetilde{M}}(n,m)$ is then defined by

$$\mathscr{H}_{\widetilde{M}}(n,m) = \{n_{\widetilde{U}_\alpha} | \alpha \in \widetilde{I}\}.$$

Notice that $\widetilde{M}(n_1, \cdots, n_m)$ is smoothly. We know that $\mathscr{H}_{\widetilde{M}}(n,m)$ is finite. This set is important to the definition of integral and the establishing of Stokes' or Gauss' theorems on smoothly combinatorial manifolds. We characterize it by a combinatorial manner in this section.

A *vertex-edge labeled graph* $G([1,k],[1,l])$ is a connected graph $G = (V, E)$ with two mappings

$$\tau_1 : V \to \{1, 2, \cdots, k\},$$

$$\tau_2 : E \to \{1, 2, \cdots, l\}$$

for integers k and l. For example, two vertex-edge labeled graphs with an underlying graph K_4 are shown in Fig.3.1.

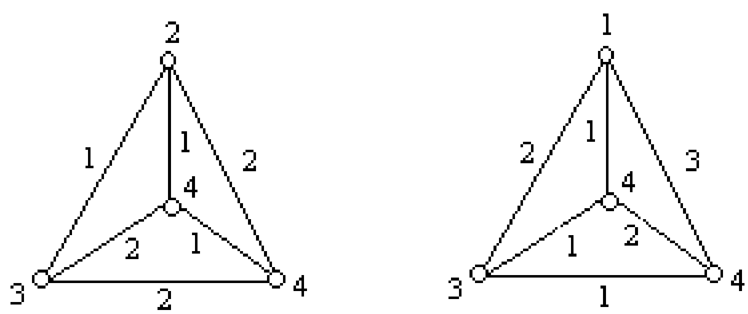

Fig.3.1

For a combinatorial finite manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ with $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$, there is a natural $1-1$ mapping $\theta : \widetilde{M}(n_1, n_2, \cdots, n_m) \to G([0, n_m], [0, n_m])$ determined in the following. Define

$$V(G([0,n_m],[0,n_m])) = V_1 \bigcup V_2,$$

where $V_1 = \{n_i -$ manifolds M^{n_i} in $\widetilde{M}(n_1,n_2,\cdots,n_m)|1 \leq i \leq m\}$ and $V_2 = \{$isolated inters$-$ection points $O_{M^{n_i},M^{n_j}}$ ofM^{n_i}, M^{n_j} in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ for $1 \leq i,j \leq m\}$, and label each n_i-manifold M^{n_i} in V_1 or O in V_2 by $\tau_1(M^{n_i}) = n_i$, $\tau_1(O) = 0$. Choose

$$E(G([0,n_m],[0,n_m])) = E_1 \bigcup E_2,$$

where $E_1 = \{(M^{n_i}, M^{n_j})|\dim(M^{n_i} \bigcap M^{n_j}) \geq 1, 1 \leq i,j \leq m\}$ and $E_2 = \{(O_{M^{n_i},M^{n_j}}, M^{n_i}),$ $(O_{M^{n_i},M^{n_j}}, M^{n_j})|M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i},M^{n_j}}$ for $1 \leq i,j \leq m\}$, and for an edge $(M^{n_i}, M^{n_j}) \in E_1$ or $(O_{M^{n_i},M^{n_j}}, M^{n_i}) \in E_2$, label it by $\tau_2(M^{n_i}, M^{n_j}) = \dim(M^{n_i} \bigcap M^{n_j})$ or 0, respectively. This construction then enables us getting a $1-1$ mapping $\theta : \widetilde{M}(n_1,n_2,\cdots,n_m) \rightarrow G([0,n_m],[0,n_m])$.

Now let $\mathcal{H}(n_1,n_2,\cdots,n_m)$ denote all finitely combinatorial manifolds $\widetilde{M}(n_1,n_2,\cdots,n_m)$ and let $\mathcal{G}[0,n_m]$ denote all vertex-edge labeled graphs $G([0,n_m],[0,n_m])$ with conditions following hold.

(1) Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2) For each edge $e = (u,v) \in E(G)$, $\tau_2(e) \leq \min\{\tau_1(u), \tau_1(v)\}$.

Then we know a relation between sets $\mathcal{H}(n_1,n_2,\cdots,n_m)$ and $\mathcal{G}([0,n_m],[0,n_m])$.

Theorem 3.1 *Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ defines a vertex-edge labeled graph $G([0,n_m],[0,n_m]) \in \mathcal{G}[0,n_m]$. Conversely, every vertex-edge labeled graph $G([0,n_m],[0,n_m]) \in \mathcal{G}[0,n_m]$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ with a $1-1$ mapping $\theta : G([0,n_m],[0,n_m]) \rightarrow \widetilde{M}$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v,w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0,n_m],[0,n_m]))$ and $\forall (v,w) \in E(G([0,n_m],[0,n_m]))$.*

Proof By definition, for $\forall \widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ there is a vertex-edge labeled graph $G([0,n_m],[0,n_m]) \in \mathcal{G}([0,n_m],[0,n_m])$ and a $1-1$ mapping $\theta : \widetilde{M} \rightarrow G([0,n_m],[0,n_m])$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$. For completing the proof, we need to construct a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ for $\forall G([0,n_m],[0,n_m]) \in \mathcal{G}[0,n_m]$ with $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v,w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0,n_m],[0,n_m]))$ and $\forall (v,w) \in E(G([0,n_m],[0,n_m]))$. The construction is carried out by programming following.

STEP 1. Choose $|G([0,n_m],[0,n_m])| - |V_0|$ manifolds correspondent to each vertex u with a dimensional n_i if $\tau_1(u) = n_i$, where $V_0 = \{u|u \in V(G([0,n_m],[0,n_m]))$ and $\tau_1(u) = 0\}$. Denoted by $V_{\geq 1}$ all these vertices in $G([0,n_m],[0,n_m])$ with label≥ 1.

STEP 2. For $\forall u_1 \in V_{\geq 1}$ with $\tau_1(u_1) = n_{i_1}$, if its neighborhood set $N_{G([0,n_m],[0,n_m])}(u_1) \bigcap V_{\geq 1} = \{v_1^1, v_1^2, \cdots, v_1^{s(u_1)}\}$ with $\tau_1(v_1^1) = n_{11}$, $\tau_1(v_1^2) = n_{12}$, $\cdots$, $\tau_1(v_1^{s(u_1)}) = n_{1s(u_1)}$, then let the manifold correspondent to the vertex u_1 with an intersection dimension $\tau_2(u_1 v_1^i)$ with manifold correspondent to the vertex v_1^i for $1 \leq i \leq s(u_1)$ and define a vertex set $\Delta_1 = \{u_1\}$.

STEP 3. If the vertex set $\Delta_l = \{u_1, u_2, \cdots, u_l\} \subseteq V_{\geq 1}$ has been defined and $V_{\geq 1} \setminus \Delta_l \neq \emptyset$, let $u_{l+1} \in V_{\geq 1} \setminus \Delta_l$ with a label $n_{i_{l+1}}$. Assume

$$(N_{G([0,n_m],[0,n_m])}(u_{l+1}) \bigcap V_{\geq 1}) \setminus \Delta_l = \{v_{l+1}^1, v_{l+1}^2, \cdots, v_{l+1}^{s(u_{l+1})}\}$$

with $\tau_1(v_{l+1}^1) = n_{l+1,1}$, $\tau_1(v_{l+1}^2) = n_{l+1,2}$, $\cdots$,$\tau_1(v_{l+1}^{s(u_{l+1})}) = n_{l+1,s(u_{l+1})}$. Then let the manifold correspondent to the vertex u_{l+1} with an intersection dimension $\tau_2(u_{l+1}v_{l+1}^i)$ with the manifold correspondent to the vertex $v_{l+1}^i, 1 \leq i \leq s(u_{l+1})$ and define a vertex set $\Delta_{l+1} = \Delta_l \bigcup \{u_{l+1}\}$.

STEP 4. Repeat steps 2 and 3 until a vertex set $\Delta_t = V_{\geq 1}$ has been constructed. This construction is ended if there are no vertices $w \in V(G)$ with $\tau_1(w) = 0$, i.e., $V_{\geq 1} = V(G)$. Otherwise, go to the next step.

STEP 5. For $\forall w \in V(G([0,n_m],[0,n_m])) \setminus V_{\geq 1}$, assume $N_{G([0,n_m],[0,n_m])}(w) = \{w_1, w_2, \cdots, w_e\}$. Let all these manifolds correspondent to vertices $w_1, w_2, \cdots, w_e$ intersects at one point simultaneously and define a vertex set $\Delta_{t+1}^* = \Delta_t \bigcup \{w\}$.

STEP 6. Repeat STEP 5 for vertices in $V(G([0,n_m],[0,n_m])) \setminus V_{\geq 1}$. This construction is finally ended until a vertex set $\Delta_{t+h}^* = V(G[n_1, n_2, \cdots, n_m])$ has been constructed.

A finitely combinatorial manifold $\widetilde{M}$ correspondent to $G([0,n_m],[0,n_m])$ is gotten when Δ_{t+h}^* has been constructed. By this construction, it is easily verified that $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ with $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v,w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0,n_m],[0,n_m]))$ and $\forall (v,w) \in E(G([0,n_m],[0,n_m]))$. This completes the proof. $\square$

Now we determine the integer set $\mathscr{H}_{\widetilde{M}}(n,m)$ for a given smoothly combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$. Notice the relation between sets $\mathcal{H}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}([0,n_m],[0,n_m])$ established in Theorem 2.4. We can determine it under its vertex-edge labeled graph $G([0,n_m],$ $[0,n_m])$.

Theorem 3.2 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with a correspondent vertex-edge labeled graph $G([0,n_m],[0,n_m])$. Then*

$$\begin{aligned}\mathscr{H}_{\widetilde{M}}(n,m) \subseteq \quad & \{n_1, n_2, \cdots, n_m\} \bigcup_{\widehat{d}(p) \geq 3, p \in \widetilde{M}} \{\widehat{d}(p) + \sum_{i=1}^{d(p)} (n_i - \widehat{d}(p))\} \\ & \bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}.\end{aligned}$$

Particularly, if $G([0,n_m],[0,n_m])$ is K_3-free, then

$$\begin{aligned}\mathscr{H}_{\widetilde{M}}(n,m) = \quad & \{\tau_1(u) | u \in V(G([0,n_m],[0,n_m]))\} \\ & \bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}.\end{aligned}$$

Proof Notice that the dimension of a point $p \in \widetilde{M}$ is

$$n_p = \widehat{d}(p) + \sum_{i=1}^{d(p)} (n_i - \widehat{d}(p))$$

by definition. If $d(p) = 1$, then $n_p = n_j, 1 \leq j \leq m$. If $d(p) = 2$, namely, $p \in M^{n_i} \cap M^{n_j}$ for $1 \leq i, j \leq m$, we know that its dimension is

$$n_i + n_j - \widehat{d}(p) = \tau_1(M^{n_i}) + \tau_1(M^{n_j}) - \widehat{d}(p).$$

Whence, we get that

$$\begin{aligned} \mathscr{H}_{\widetilde{M}}(n,m) \subseteq \quad & \{n_1, n_2, \cdots, n_m\} \bigcup_{\widehat{d}(p) \geq 3, p \in \widetilde{M}} \{\widehat{d}(p) + \sum_{i=1}^{d(p)} (n_i - \widehat{d}(p))\} \\ & \bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}. \end{aligned}$$

Now if $G([0,n_m],[0,n_m])$ is K_3-free, then there are no points with intersectional dimension$\geq$ 3. In this case, there are really existing points $p \in M^{n_i}$ for any integer $i, 1 \leq i \leq m$ and $q \in M^{n_i} \cap M^{n_j}$ for $1 \leq i, j \leq m$ by definition. Therefore, we get that

$$\begin{aligned} \mathscr{H}_{\widetilde{M}}(n,m) = \quad & \{\tau_1(u) | u \in V(G([0,n_m],[0,n_m]))\} \\ & \bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}. \square \end{aligned}$$

For some special graphs, we get the following interesting results for the integer set $\mathscr{H}_{\widetilde{M}}(n,m)$.

Corollary 3.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with a correspondent vertex-edge labeled graph $G([0,n_m],[0,n_m])$. If $G([0,n_m],[0,n_m]) \cong P^s$, then*

$$\mathscr{H}_{\widetilde{M}}(n,m) = \{\tau_1(u_i), 1 \leq i \leq p\} \bigcup \{\tau_1(u_i) + \tau_1(u_{i+1}) - \tau_2(u_i, u_{i+1}) | 1 \leq i \leq p-1\}$$

and if $G([0,n_m],[0,n_m]) \cong C^p$ with $p \geq 4$, then

$$\mathscr{H}_{\widetilde{M}}(n,m) = \{\tau_1(u_i), 1 \leq i \leq p\} \bigcup \{\tau_1(u_i) + \tau_1(u_{i+1}) - \tau_2(u_i, u_{i+1}) | 1 \leq i \leq p, i \equiv (modp)\}.$$

§4. Integration on combinatorial manifolds

We generalize the integration on manifolds to combinatorial manifolds and show it is independent on the choice of local charts and partition of unity in this section.

4.1 Partition of unity

Definition 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $\omega \in \Lambda(\widetilde{M})$. A support set* Supp$\omega$ *of ω is defined by*

$$\mathrm{Supp}\omega = \overline{\{p \in \widetilde{M}; \omega(p) \neq 0\}}$$

and say ω has compact support if Suppω *is compact in $\widetilde{M}$. A collection of subsets $\{C_i | i \in \widetilde{I}\}$ of $\widetilde{M}$ is called locally finite if for each $p \in \widetilde{M}$, there is a neighborhood U_p of p such that $U_p \cap C_i = \emptyset$ except for finitely many indices i.*

A *partition of unity* on a combinatorial manifold $\widetilde{M}$ is defined in the next.

Definition 4.2 *A partition of unity on a combinatorial manifold $\widetilde{M}$ is a collection $\{(U_i, g_i) | i \in \widetilde{I}\}$, where*

(1) $\{U_i | i \in \widetilde{I}\}$ is a locally finite open covering of $\widetilde{M}$;
(2) $g_i \in \mathscr{X}(\widetilde{M})$, $g_i(p) \geq 0$ for $\forall p \in \widetilde{M}$ and supp$g_i \in U_i$ for $i \in \widetilde{I}$;
(3) For $p \in \widetilde{M}$, $\sum\limits_i g_i(p) = 1$.

For a smoothly combinatorial manifold $\widetilde{M}$, denoted by $G[\widetilde{M}]$ the underlying graph of its correspondent vertex-edge labeled graph. We get the next result for a partition of unity on smoothly combinatorial manifolds.

Theorem 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then $\widetilde{M}$ admits partitions of unity.*

Proof For $\forall M \in V(G[\widetilde{M}])$, since $\widetilde{M}$ is smooth we know that M is a smoothly submanifold of $\widetilde{M}$. As a byproduct, there is a partition of unity $\{(U_M^\alpha, g_M^\alpha) | \alpha \in I_M\}$ on M with conditions following hold.

(1) $\{U_M^\alpha | \alpha \in I_M\}$ is a locally finite open covering of M;

(2) $g_M^\alpha(p) \geq 0$ for $\forall p \in M$ and supp$g_M^\alpha \in U_M^\alpha$ for $\alpha \in I_M$;

(3) For $p \in M$, $\sum\limits_i g_M^i(p) = 1$.

By definition, for $\forall p \in \widetilde{M}$, there is a local chart $(U_p, [\varphi_p])$ enable $\varphi_p : U_p \to B^{n_{i_1}} \bigcup B^{n_{i_2}} \bigcup \cdots \bigcup B^{n_{i_{s(p)}}}$ with $B^{n_{i_1}} \bigcap B^{n_{i_2}} \bigcap \cdots \bigcap B^{n_{i_{s(p)}}} \neq \emptyset$. Now let $U_{M_{i_1}}^\alpha, U_{M_{i_2}}^\alpha, \cdots, U_{M_{i_{s(p)}}}^\alpha$ be $s(p)$ open sets on manifolds $M, M \in V(G[\widetilde{M}])$ such that

$$p \in U_p^\alpha = \bigcup_{h=1}^{s(p)} U_{M_{i_h}}^\alpha. \quad (4.1)$$

We define

$$\widetilde{S}(p) = \{U_p^\alpha | \text{ all integers } \alpha \text{ enabling (4.1) hold}\}.$$

Then

$$\widetilde{\mathcal{A}} = \bigcup_{p \in \widetilde{M}} \widetilde{S}(p) = \{U_p^\alpha | \alpha \in \widetilde{I}(p)\}$$

is locally finite covering of the combinatorial manifold $\widetilde{M}$ by properties $(1)-(3)$. For $\forall U_p^{\alpha} \in \widetilde{S}(p)$, define

$$\sigma_{U_p^{\alpha}} = \sum_{s\geq 1} \sum_{\{i_1,i_2,\cdots,i_s\}\subset\{1,2,\cdots,s(p)\}} (\prod_{h=1}^{s} g_{M_{i_h}^{\varsigma}})$$

and

$$g_{U_p^{\alpha}} = \frac{\sigma_{U_p^{\alpha}}}{\sum\limits_{\widetilde{V}\in\widetilde{S}(p)} \sigma_{\widetilde{V}}}.$$

Then it can be checked immediately that $\{(U_p^{\alpha}, g_{U_p^{\alpha}})|p \in \widetilde{M}, \alpha \in \widetilde{I}(p)\}$ is a partition of unity on $\widetilde{M}$ by properties (1)-(3) on g_M^{α} and the definition of $g_{U_p^{\alpha}}$. $\square$

Corollary 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an atlas $\widetilde{\mathcal{A}} = \{(V_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ and t_α be a C^k tensor field, $k \geq 1$, of field type (r,s) defined on V_α for each α, and assume that there exists a partition of unity $\{(U_i, g_i)|i \in J\}$ subordinate to $\widetilde{\mathcal{A}}$, i.e., for $\forall i \in J$, there exists $\alpha(i)$ such that $U_i \subset V_{\alpha(i)}$. Then for $\forall p \in \widetilde{M}$,*

$$t(p) = \sum_i g_i t_{\alpha(i)}$$

is a C^k tensor field of type (r,s) on $\widetilde{M}$

Proof Since $\{U_i|i \in J\}$ is locally finite, the sum at each point p is a finite sum and $t(p)$ is a type (r,s) for every $p \in \widetilde{M}$. Notice that t is C^k since the local form of t in a local chart $(V_{\alpha(i)}, [\varphi_{\alpha(i)}])$ is

$$\sum_j g_i t_{\alpha(j)},$$

where the summation taken over all indices j such that $V_{\alpha(i)} \bigcap V_{\alpha(j)} \neq \emptyset$. Those number j is finite by the local finiteness. $\square$

4.2 **Integration on combinatorial manifolds**

First, we introduce integration on combinatorial Euclidean spaces. Let $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$ be a combinatorially Euclidean space and

$$\tau : \widetilde{\mathbf{R}}(n_1,\cdots,n_m) \to \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$$

a C^1 differential mapping with

$$[\overline{y}] = [y^{\kappa\lambda}]_{m\times n_m} = [\tau^{\kappa\lambda}([x^{\mu\nu}])]_{m\times n_m}.$$

The *Jacobi matrix* of f is defined by

$$\frac{\partial[\overline{y}]}{\partial[\overline{x}]} = [A_{(\kappa\lambda)(\mu\nu)}],$$

where $A_{(\kappa\lambda)(\mu\nu)} = \frac{\partial \tau^{\kappa\lambda}}{\partial x^{\mu\nu}}$.

Now let $\omega \in T_k^0(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$, a *pull-back* $\tau^*\omega \in T_k^0(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is defined by

$$\tau^*\omega(a_1, a_2, \cdots, a_k) = \omega(f(a_1), f(a_2), \cdots, f(a_k))$$

for $\forall a_1, a_2, \cdots, a_k \in \widetilde{R}$.

Denoted by $n = \widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m})$. If $0 \leq l \leq n$, recall([4]) that the basis of $\Lambda^l(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is

$$\{\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \cdots \wedge \mathbf{e}^{i_l} | 1 \leq i_1 < i_2 \cdots < i_l \leq n\}$$

for a basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ and its dual basis $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n$. Thereby the dimension of $\Lambda^l(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is

$$\begin{pmatrix} n \\ l \end{pmatrix} = \frac{(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}))!}{l!(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}) - l)!}.$$

Whence $\Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is one-dimensional. Now if ω_0 is a basis of $\Lambda^n(\widetilde{R})$, we then know that its each element ω can be represented by $\omega = c\omega_0$ for a number $c \in \mathbf{R}$. Let $\tau : \widetilde{\mathbf{R}}(n_1, \cdots, n_m) \to \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ be a linear mapping. Then

$$\tau^* : \Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m)) \to \Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$$

is also a linear mapping with $\tau^*\omega = c\tau^*\omega_0 = b\omega$ for a unique constant $b = \det\tau$, called the determinant of τ. It has been known that ([1])

$$\det\tau = \det(\frac{\partial[\overline{y}]}{\partial[\overline{x}]})$$

for a given basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ and its dual basis $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n$.

Definition 4.3 *Let $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ be a combinatorial Euclidean space,$n = \widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m})$, $\widetilde{U} \subset \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ and $\omega \in \Lambda^n(U)$ have compact support with*

$$\omega(x) = \omega_{(\mu_{i_1}\nu_{i_1})\cdots(\mu_{i_n}\nu_{i_n})} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_n}\nu_{i_n}}$$

relative to the standard basis $\mathbf{e}^{\mu\nu}, 1 \leq \mu \leq m, 1 \leq \nu \leq n_m$ of $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ with $\mathbf{e}^{\mu\nu} = e^{\nu}$ for $1 \leq \mu \leq \widehat{m}$. An integral of ω on $\widetilde{U}$ is defined to be a mapping $\int_{\widetilde{U}} : f \to \int_{\widetilde{U}} f \in \mathbf{R}$ with

$$\int_{\widetilde{U}} \omega = \int \omega(x) \prod_{\nu=1}^{\widehat{m}} dx^{\nu} \prod_{\mu \geq \widehat{m}+1, 1 \leq \nu \leq n_i} dx^{\mu\nu}, \qquad (4.2)$$

where the right hand side of (4.2) *is the Riemannian integral of ω on $\widetilde{U}$.*

For example, consider the combinatorial Euclidean space $\widetilde{\mathbf{R}}(3,5)$ with $\mathbf{R}^3 \cap \mathbf{R}^5 = \mathbf{R}$. Then the integration of an $\omega \in \Lambda^7(\widetilde{U})$ for an open subset $\widetilde{U} \in \widetilde{\mathbf{R}}(3,5)$ is

$$\int_{\widetilde{U}}\omega=\int_{\widetilde{U}\cap(\mathbf{R}^3\cup\mathbf{R}^5)}\omega(x)dx^1dx^{12}dx^{13}dx^{22}dx^{23}dx^{24}dx^{25}.$$

Theorem 4.2 *Let U and V be open subsets of $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$ and $\tau:U\rightarrow V$ is an orientation-preserving diffeomorphism. If $\omega\in\Lambda^n(V)$ has a compact support for $n=\widehat{m}+\sum\limits_{i=1}^{m}(n_i-\widehat{m})$, then $\tau^*\omega\in\Lambda^n(U)$ has compact support and*

$$\int\tau^*\omega=\int\omega.$$

Proof Let $\omega(x)=\omega_{(\mu_{i_1}\nu_{i_1})\cdots(\mu_{i_n}\nu_{i_n})}dx^{\mu_{i_1}\nu_{i_1}}\wedge\cdots\wedge dx^{\mu_{i_n}\nu_{i_n}}\in\Lambda^n(V)$. Since τ is a diffeomorphism, the support of $\tau^*\omega$ is $\tau^{-1}(\mathrm{supp}\omega)$, which is compact by that of $\mathrm{supp}\omega$ compact.

By the usual change of variables formula, since $\tau^*\omega=(\omega\circ\tau)(\det\tau)\omega_0$ by definition, where $\omega_0=dx^1\wedge\cdots\wedge dx^{\widehat{m}}\wedge dx^{1(\widehat{m}+1)}\wedge dx^{1(\widehat{m}+2)}\wedge\cdots\wedge dx^{1n_1}\wedge\cdots\wedge dx^{mn_m}$, we then get that

$$\begin{aligned}\int\tau^*\omega &= \int(\omega\circ\tau)(\det\tau)\prod_{\nu=1}^{\widehat{m}}dx^{\nu}\prod_{\mu\geq\widehat{m}+1,1\leq\nu\leq n_\mu}dx^{\mu\nu}\\ &= \int\omega.\end{aligned}$$

□

Definition 4.4 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. If there exists a family $\{(U_\alpha,[\varphi_\alpha]|\alpha\in\widetilde{I})\}$ of local charts such that*

(1) $\bigcup\limits_{\alpha\in\widetilde{I}}U_\alpha=\widetilde{M}$;

(2) *for $\forall\alpha,\beta\in\widetilde{I}$, either $U_\alpha\bigcap U_\beta=\emptyset$ or $U_\alpha\bigcap U_\beta\neq\emptyset$ but for $\forall p\in U_\alpha\bigcap U_\beta$, the Jacobi matrix*

$$det(\frac{\partial[\varphi_\beta]}{\partial[\varphi_\alpha]})>0,$$

then $\widetilde{M}$ is called an oriently combinatorial manifold and $(U_\alpha,[\varphi_\alpha])$ an oriented chart for $\forall\alpha\in\widetilde{I}$.

Now for any integer $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$, we can define an integral of $\widetilde{n}$-forms on a smoothly combinatorial manifold $\widetilde{M}(n_1,\cdots,n_m)$.

Definition 4.5 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with orientation $\mathscr{O}$ and $(\widetilde{U};[\varphi])$ a positively oriented chart with a constant $n_{\widetilde{U}}\in\mathscr{H}_{\widetilde{M}}(n,m)$. Suppose $\omega\in\Lambda^{n_{\widetilde{U}}}(\widetilde{M}),\widetilde{U}\subset\widetilde{M}$ has compact support $\widetilde{C}\subset\widetilde{U}$. Then define*

$$\int_{\widetilde{C}}\omega=\int\varphi_*(\omega|_{\widetilde{U}}).\qquad(4.3)$$

Now if $\mathscr{C}_{\widetilde{M}}$ is an atlas of positively oriented charts with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$, let $\widetilde{P}=\{(\widetilde{U}_\alpha,\varphi_\alpha,g_\alpha)|\alpha\in\widetilde{I}\}$ be a partition of unity subordinate to $\mathscr{C}_{\widetilde{M}}$. For $\forall\omega\in\Lambda^{\widetilde{n}}(\widetilde{M})$, $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$, an integral of ω on $\widetilde{P}$ is defined by

$$\int_{\widetilde{P}} \omega = \sum_{\alpha \in \widetilde{I}} \int g_\alpha \omega. \qquad (4.4)$$

The next result shows that the integral of $\widetilde{n}$-forms for $\forall \widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n,m)$ is well-defined.

Theorem 4.3 *Let $\widetilde{M}(n_1, \cdots, n_m)$ be a smoothly combinatorial manifold. For $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n,m)$, the integral of $\widetilde{n}$-forms on $\widetilde{M}(n_1, \cdots, n_m)$ is well-defined, namely, the sum on the right hand side of* (4.4) *contains only a finite number of nonzero terms, not dependent on the choice of $\mathscr{C}_{\widetilde{M}}$ and if P and Q are two partitions of unity subordinate to $\mathscr{C}_{\widetilde{M}}$, then*

$$\int_{\widetilde{P}} \omega = \int_{\widetilde{Q}} \omega.$$

Proof By definition for any point $p \in \widetilde{M}(n_1, \cdots, n_m)$, there is a neighborhood $\widetilde{U}_p$ such that only a finite number of g_α are nonzero on $\widetilde{U}_p$. Now by the compactness of suppω, only a finite number of such neighborhood cover suppω. Therefore, only a finite number of g_α are nonzero on the union of these $\widetilde{U}_p$, namely, the sum on the right hand side of (4.4) contains only a finite number of nonzero terms.

Notice that the integral of $\widetilde{n}$-forms on a smoothly combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$ is well-defined for a local chart $\widetilde{U}$ with a constant $n_{\widetilde{U}} = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ for $\forall p \in \widetilde{U} \subset \widetilde{M}(n_1, \cdots, n_m)$ by (4.3) and Definition 4.3. Whence each term on the right hand side of (4.4) is well-defined. Thereby $\int_{\widetilde{P}} \omega$ is well-defined.

Now let $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha) | \alpha \in \widetilde{I}\}$ and $\widetilde{Q} = \{(\widetilde{V}_\beta, \varphi_\beta, h_\beta) | \beta \in \widetilde{J}\}$ be partitions of unity subordinate to atlas $\mathscr{C}_{\widetilde{M}}$ and $\mathscr{C}^*_{\widetilde{M}}$ with respective integer sets $\mathscr{H}_{\widetilde{M}}(n,m)$ and $\mathscr{H}^*_{\widetilde{M}}(n,m)$. Then these functions $\{g_\alpha h_\beta\}$ satisfy $g_\alpha h_\beta(p) = 0$ except only for a finite number of index pairs (α, β) and

$$\sum_\alpha \sum_\beta g_\alpha h_\beta(p) = 1, \quad \text{for } \forall p \in \widetilde{M}(n_1, \cdots, n_m).$$

Since $\sum\limits_\beta = 1$, we then get that

$$\begin{aligned} \int_{\widetilde{P}} &= \sum_\alpha \int g_\alpha \omega = \sum_\beta \sum_\alpha \int h_\beta g_\alpha \omega \\ &= \sum_\alpha \sum_\beta \int g_\alpha h_\beta \omega = \int_{\widetilde{Q}} \omega. \end{aligned}$$

□

By the relation of smoothly combinatorial manifolds with these vertex-edge labeled graphs established in Theorem 3.1, we can also get the integration on a vertex-edge labeled graph $G([0, n_m], [0, n_m])$ by viewing it that of the correspondent smoothly combinatorial manifold $\widetilde{M}$ with $\Lambda^l(G) = \Lambda^l(\widetilde{M})$, $\mathscr{H}_G(n,m) = \mathscr{H}_{\widetilde{M}}(n,m)$, namely define the *integral of an $\widetilde{n}$-form ω on $G([0, n_m], [0, n_m])$ for $\widetilde{n} \in \mathscr{H}_G(n,m)$* by

$$\int_{G([0,n_m],[0,n_m])} \omega = \int_{\widetilde{M}} \omega.$$

Then each result in this paper can be restated by combinatorial words, such as Theorem 5.1 and its corollaries in next section.

Now let $n_1, n_2, \cdots, n_m$ be a positive integer sequence. For any point $p \in \widetilde{M}$, if there is a local chart $(\widetilde{U}_p, [\varphi_p])$ such that $[\varphi_p] : U_p \to B^{n_1} \bigcup B^{n_2} \bigcup \cdots \bigcup B^{n_m}$ with $\dim(B^{n_1} \bigcap B^{n_2} \bigcap \cdots \bigcap B^{n_m}) = \widehat{m}$, then $\widetilde{M}$ is called a *homogenously combinatorial manifold*. Particularly, if $m = 1$, a homogenously combinatorial manifold is nothing but a manifold. We then get consequences for the integral of $(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}))$-forms on homogenously combinatorial manifolds.

Corollary 4.2 *The integral of $(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}))$-forms on a homogenously combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is well-defined, particularly, the integral of n-forms on an n-manifold is well-defined.*

Similar to Theorem 4.2 for the *change of variables formula of integral* in a combinatorial Euclidean space, we get that of formula in smoothly combinatorial manifolds.

Theorem 4.4 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\widetilde{N}(k_1, k_2, \cdots, k_l)$ be oriently combinatorial manifolds and $\tau : \widetilde{M} \to \widetilde{N}$ an orientation-preserving diffeomorphism. If $\omega \in \Lambda^{\widetilde{k}}(\widetilde{N})$, $\widetilde{k} \in \mathscr{H}_{\widetilde{N}}(k, l)$ has compact support, then $\tau^*\omega$ has compact support and*

$$\int \omega = \int \tau^*\omega.$$

Proof Notice that $\mathrm{supp}\tau^*\omega = \tau^{-1}(\mathrm{supp}\omega)$. Thereby $\tau^*\omega$ has compact support since ω has so. Now let $\{(U_i, \varphi_i) | i \in \widetilde{I}\}$ be an atlas of positively oriented charts of $\widetilde{M}$ and $\widetilde{P} = \{g_i | i \in \widetilde{I}\}$ a subordinate partition of unity with an integer set $\mathscr{H}_{\widetilde{M}}(n, m)$. Then $\{(\tau(U_i), \varphi_i \circ \tau^{-1}) | i \in \widetilde{I}\}$ is an atlas of positively oriented charts of $\widetilde{N}$ and $\widetilde{Q} = \{g_i \circ \tau^{-1}\}$ is a partition of unity subordinate to the covering $\{\tau(U_i) | i \in \widetilde{I}\}$ with an integer set $\mathscr{H}_{\tau(\widetilde{M})}(k, l)$. Whence, we get that

$$\begin{aligned}
\int \tau^*\omega &= \sum_i \int g_i \tau^*\omega = \sum_i \int \varphi_{i*}(g_i \tau^*\omega) \\
&= \sum_i \int \varphi_{i*}(\tau^{-1})_*(g_i \circ \tau^{-1})\omega = \sum_i \int (\varphi_i \circ \tau^{-1})_*(g_i \circ \tau^{-1})\omega = \int \omega.
\end{aligned}$$

□

§5. A generalized of Stokes' or Gauss' theorem

Definition 5.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. A subset $\widetilde{D}$ of $\widetilde{M}$ is with boundary if its points can be classified into two classes following.*

Class 1**(interior point Int**$\widetilde{D}$**)** *For* $\forall p \in \mathrm{Int}D$*, there is a neighborhood* $\widetilde{V}_p$ *of* p *enable* $\widetilde{V}_p \subset \widetilde{D}$*.*

Case 2**(boundary** $\partial\widetilde{D}$**)** *For* $\forall p \in \partial\widetilde{D}$*, there is integers* μ, ν *for a local chart* $(U_p; [\varphi_p])$ *of* p *such that* $x^{\mu\nu}(p) = 0$ *but*

$$\widetilde{U}_p \cap \widetilde{D} = \{q | q \in U_p, x^{\kappa\lambda} \geq 0 \ for \ \forall\{\kappa, \lambda\} \neq \{\mu, \nu\}\}.$$

Then we generalize the famous Stokes' theorem on manifolds in the next.

Theorem 5.1 *Let* $\widetilde{M}$ *be a smoothly combinatorial manifold with an integer set* $\mathscr{H}_{\widetilde{M}}(n, m)$ *and* $\widetilde{D}$ *a boundary subset of* $\widetilde{M}$*. For* $\forall\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n, m)$ *if* $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ *has a compact support, then*

$$\int_{\widetilde{D}} \widetilde{d}\omega = \int_{\partial\widetilde{D}} \omega$$

with the convention $\int_{\partial\widetilde{D}} \omega = 0$ *while* $\partial\widetilde{D} = \emptyset$*.*

Proof By Definition 4.5, the integration on a smoothly combinatorial manifold was constructed with partitions of unity subordinate to an atlas. Let $\mathscr{C}_{\widetilde{M}}$ be an atlas of positively oriented charts with an integer set $\mathscr{H}_{\widetilde{M}}(n, m)$ and $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha) | \alpha \in \widetilde{I}\}$ a partition of unity subordinate to $\mathscr{C}_{\widetilde{M}}$. Since supp$\omega$ is compact, we know that

$$\int_{\widetilde{D}} \widetilde{d}\omega = \sum_{\alpha \in \widetilde{I}} \int_{\widetilde{D}} \widetilde{d}(g_\alpha\omega),$$

$$\int_{\partial\widetilde{D}} \omega = \sum_{\alpha \in \widetilde{I}} \int_{\partial\widetilde{D}} g_\alpha\omega.$$

and there are only finite nonzero terms on the right hand side of the above two formulae. Thereby, we only need to prove

$$\int_{\widetilde{D}} \widetilde{d}(g_\alpha\omega) = \int_{\partial\widetilde{D}} g_\alpha\omega$$

for $\forall\alpha \in \widetilde{I}$.

Not loss of generality we can assume that ω is an $\widetilde{n}$-forms on a local chart $(\widetilde{U}, [\varphi])$ with a compact support for $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n, m)$. Now write

$$\omega = \sum_{h=1}^{\widetilde{n}} (-1)^{h-1} \omega_{\mu_{i_h}\nu_{i_h}} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge \widehat{dx^{\mu_{i_h}\nu_{i_h}}} \wedge \cdots \wedge dx^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}},$$

where $\widehat{dx^{\mu_{i_h}\nu_{i_h}}}$ means that $dx^{\mu_{i_h}\nu_{i_h}}$ is deleted, where

$$i_h \in \{1, \cdots, \widehat{n}_U, (1(\widehat{n}_U + 1)), \cdots, (1n_1), (2(\widehat{n}_U + 1)), \cdots, (2n_2), \cdots, (mn_m)\}.$$

Then

$$\widetilde{d}\omega = \sum_{h=1}^{\widetilde{n}} \frac{\partial \omega_{\mu_{i_h}\nu_{i_h}}}{\partial x^{\mu_{i_h}\nu_{i_h}}} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}. \quad (5.1)$$

Consider the appearance of neighborhood $\widetilde{U}$. There are two cases must be considered.

Case 1 $\widetilde{U} \bigcap \partial\widetilde{D} = \emptyset$

In this case, $\int_{\partial\widetilde{D}} \omega = 0$ and $\widetilde{U}$ is in $\widetilde{M} \setminus \widetilde{D}$ or in $\mathbf{Int}\widetilde{D}$. The former is naturally implies that $\int_{\widetilde{D}} \widetilde{d}(g_\alpha\omega) = 0$. For the later, we find that

$$\int_{\widetilde{D}} \widetilde{d}\omega = \sum_{h=1}^{\widetilde{n}} \int_{\widetilde{U}} \frac{\partial \omega_{\mu_{i_h}\nu_{i_h}}}{\partial x^{\mu_{i_h}\nu_{i_h}}} dx^{\mu_{i_1}\nu_{i_1}} \cdots dx^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}. \quad (5.2)$$

Notice that $\int_{-\infty}^{+\infty} \frac{\partial \omega_{\mu_{i_h}\nu_{i_h}}}{\partial x^{\mu_{i_h}\nu_{i_h}}} dx^{\mu_{i_h}\nu_{i_h}} = 0$ since $\omega_{\mu_{i_h}\nu_{i_h}}$ has compact support. Thus $\int_{\widetilde{D}} \widetilde{d}\omega = 0$ as desired.

Case 2 $\widetilde{U} \bigcap \partial\widetilde{D} \neq \emptyset$

In this case we can do the same trick for each term except the last. Without loss of generality, assume that

$$\widetilde{U} \bigcap \widetilde{D} = \{q | q \in U, x^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}(q) \geq 0\}$$

and

$$\widetilde{U} \bigcap \partial\widetilde{D} = \{q | q \in U, x^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}(q) = 0\}.$$

Then we get that

$$\begin{aligned}
\int_{\partial\widetilde{D}} \omega &= \int_{U\cap\partial\widetilde{D}} \omega \\
&= \sum_{h=1}^{\widetilde{n}} (-1)^{h-1} \int_{U\cap\partial\widetilde{D}} \omega_{\mu_{i_h}\nu_{i_h}} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge \widehat{dx^{\mu_{i_h}\nu_{i_h}}} \wedge \cdots \wedge dx^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}} \\
&= (-1)^{\widetilde{n}-1} \int_{U\cap\partial\widetilde{D}} \omega_{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_{\widetilde{n}-1}}\nu_{i_{\widetilde{n}-1}}}
\end{aligned}$$

since $dx^{\mu_{i_n}\nu_{i_{\widetilde{n}}}}(q) = 0$ for $q \in \widetilde{U} \cap \partial\widetilde{D}$. Notice that $\mathbf{R}^{\widetilde{n}-1} = \partial\mathbf{R}^{\widetilde{n}}_+$ but the usual orientation on $\mathbf{R}^{\widetilde{n}-1}$ is not the boundary orientation, whose outward unit normal is $-\mathbf{e}_{\widetilde{n}} = (0, \cdots, 0, -1)$. Hence

$$\int_{\partial\widetilde{D}} \omega = -\int_{\partial\mathbf{R}^{\widetilde{n}}_+} \omega_{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}(x^{\mu_{i_1}\nu_{i_1}}, \cdots, x^{\mu_{i_{\widetilde{n}-1}}\nu_{i_{\widetilde{n}-1}}}, 0) dx^{\mu_{i_1}\nu_{i_1}} \cdots dx^{\mu_{i_{\widetilde{n}-1}}\nu_{i_{\widetilde{n}-1}}}.$$

On the other hand, by the fundamental theorem of calculus,

$$\begin{aligned}&\int_{\mathbf{R}^{\tilde{n}-1}}(\int_0^\infty \frac{\partial \omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}}{\partial x^{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}})dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}}\\ &=-\int_{\mathbf{R}^{\tilde{n}-1}}\omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}(x^{\mu_{i_1}\nu_{i_1}},\cdots,x^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{n-1}}\nu_{i_{n-1}}}.\end{aligned}$$

Since $\omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}$ has a compact support, thus

$$\int_U \omega = -\int_{\mathbf{R}^{\tilde{n}-1}}\omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}(x^{\mu_{i_1}\nu_{i_1}},\cdots,x^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}}.$$

Therefore, we get that

$$\int_{\widetilde{D}} \widetilde{d}\omega = \int_{\partial\widetilde{D}} \omega$$

This completes the proof. □

Corollaries following are immediately obtained by Theorem 5.1

Corollary 5.1 *Let $\widetilde{M}$ be a homogenously combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$ if $\omega\in\Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then*

$$\int_{\widetilde{D}} \widetilde{d}\omega = \int_{\partial\widetilde{D}} \omega,$$

particularly, if $\widetilde{M}$ is nothing but a manifold, the Stokes' theorem holds.

Corollary 5.2 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$. For $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$, if $\omega\in\Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then*

$$\int_{\widetilde{M}} \omega = 0.$$

By the definition of integration on vertex-edge labeled graphs $G([0,n_m],[0,n_m])$, let a boundary subset of $G([0,n_m],[0,n_m])$ mean that of its correspondent combinatorial manifold $\widetilde{M}$. Theorem 5.1 and Corollary 5.2 then can be restated by a combinatorial manner as follows.

Theorem 5.2 *Let $G([0,n_m],[0,n_m])$ be a vertex-edge labeled graph correspondent with an integer set $\mathscr{H}_G(n,m)$ and $\widetilde{D}$ a boundary subset of $G([0,n_m],[0,n_m])$. For $\forall\widetilde{n}\in\mathscr{H}_G(n,m)$ if $\omega\in\Lambda^{\widetilde{n}}(G([0,n_m],[0,n_m]))$ has a compact support, then*

$$\int_{\widetilde{D}} \widetilde{d}\omega = \int_{\partial\widetilde{D}} \omega$$

with the convention $\int_{\partial\widetilde{D}}\omega = 0$ while $\partial\widetilde{D}=\emptyset$.

Corollary 5.3 *Let $G([0,n_m],[0,n_m])$ be a vertex-edge labeled graph correspondent with an integer set $\mathscr{H}_G(n,m)$. For $\forall\widetilde{n}\in\mathscr{H}_G(n,m)$ if $\omega\in\Lambda^{\widetilde{n}}(G([0,n_m],[0,n_m]))$ has a compact support, then*

$$\int_{G([0,n_m][0,n_m])} \omega = 0.$$

Similar to the case of manifolds, we find a generalization for Gauss' theorem on smoothly combinatorial manifolds in the next.

Theorem 5.3 *Let* $\widetilde{M}$ *be a smoothly combinatorial manifold with an integer set* $\mathscr{H}_{\widetilde{M}}(n,m)$, $\widetilde{D}$ *a boundary subset of* $\widetilde{M}$ *and* $\mathbf{X}$ *a vector field on* $\widetilde{M}$ *with a compact support. Then*

$$\int_{\widetilde{D}} (\mathrm{div}\mathbf{X})\mathbf{v} = \int_{\partial\widetilde{D}} \mathbf{i_X v},$$

where $\mathbf{v}$ *is a volume form on* $\widetilde{M}$, *i.e., nonzero elements in* $\Lambda^{\widetilde{n}}(\widetilde{M})$ *for* $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n,m)$.

Proof This result is also a consequence of Theorem 5.1. Notice that

$$(\mathrm{div}\mathbf{X})\mathbf{v} = \widetilde{d}\mathbf{i_X v} + \mathbf{i_X}\widetilde{d}\mathbf{v} = \widetilde{d}\mathbf{i_X v}.$$

According to Theorem 5.1, we then get that

$$\int_{\widetilde{D}} (\mathrm{div}\mathbf{X})\mathbf{v} = \int_{\partial\widetilde{D}} \mathbf{i_X v}.$$

□

References

[1] R.Abraham, J.E.Marsden and T.Ratiu, *Manifolds, Tensor Analysis and Application*, Addison -Wesley Publishing Company, Inc, Reading, Mass, 1983.

[2] W.H.Chern and X.X.Li, *Introduction to Riemannian Geometry*, Peking University Press, 2002.

[3] L.F.Mao, Combinatorial speculations and combinatorial conjecture for mathematics, *International J. Mathematical Combinatorics*, Vol.1, No.1(2007),01-19.

[4] L.F.Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114. Also in *arXiv: math.GM/0612760*.

[5] L.F.Mao, Pseudo-Manifold Geometries with Applications, *International J.Math.Combin*, Vol.1,No.1(2007), 45-58. Also in e-print: *arXiv: math. GM/0610307*.

[6] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[7] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.

[8] F.Smarandache, Mixed non-euclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

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A Note on
the Maximum Genus of Graphs with Diameter 4

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Abstract: Let G be a simple graph with diameter four,if G does not contain complete subgraph K_3 of order three. We prove that the Betti deficient number of G, $\xi(G) \leq 2$. i.e. the maximum genus of G, $\gamma_M(G) \geq \frac{1}{2}\beta(G) - 1$ in this paper, which is related with *Smarandache 2-manifolds* with minimum faces.

Key words: Diameter, Betti deficiency number, maximum genus.

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§1. Preliminaries and known results

In this paper, G is a finite undirected simple connected graph. The maximum genus $\gamma_M(G)$ of G is the largest genus of an orientable surface on which G has a 2-cell embedding, and $\xi(G)$ is the Betti deficiency of G. To determine the maximum genus $\gamma_M(G)$ of a graph G on orientable surfaces is related with *map geometries*, i.e., *Smarandache 2-manifolds* (see [1] for details) with minimum faces.

By Xuong's theory on the maximum genus of a connected graph, $\xi(G)$ equal to $\beta(G) - 2\gamma_M(G)$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the Betti number of G. For convenience, we use *deficiency* to replace the words *Betti deficiency* in this paper. Nebeský[2] showed that if G is a connected graph and $A \subseteq E(G)$, let $\upsilon(G, A) = c(G - A) + b(G - A) - |A| - 1$, where $c(G - A)$ denotes the number of components in $G - A$ and $b(G - A)$ denotes the number of components in $G - A$ with an odd Betti number, then we have $\xi(G) = max\{\upsilon(G, A)|A \subseteq E(G)\}$.

Clearly, the maximum genus of a graph can be determined by its deficiency. In case of that $\xi(G) \leq 1$, the graph G is said to be upper embeddable. As we known, following theorems are the main results on relations of the maximum genus with diameter of a graph.

Theorem 1.1 *Let G be a multigraph of diameter 2. Then $\xi(G) \leq 1$.*

Skoviera proved Theorem 1.1 by a different method in [3] − [4].

Hunglin Fu and Minchu Tsai considered multigraphs of diameter 3 and proved the following theorem in [5].

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Theorem 1.2 *Let G be a multigraph of diameter 3. Then $\xi(G) \leq 2$.*

When the diameter of graphs is larger than 3, the Betti deficiency of G is unbounded. The following investigations have focused on graphs with a given diameter and some characters. Some results in this direction are presented in the following.

Theorem 1.3([16]) *Let G be a 3-connected multigraph of diameter 4, then $\xi(G) \leq 4$.*

Theorem 1.4([16]) *Let G be a 3-connected simple graph of diameter 5. Then $\xi(G) \leq 18$.*

Yuanqiu Huang and Yanpei Liu proved the following result in [6].

Theorem 1.5 *Let G be a simple, K_3-free graph of diameter 4, then $\xi(G) \leq 4$, where K_3-free graph means that there are no spanning subgraphs K_3 in G.*

The main purpose of this paper is to improve this result.

§2. Main result and its proof

Nebeský's method is useful and the minimality property of the edge subset A in this method plays an important role. For convenience, we call a graph with $\xi(G) \geq 2$ a *deficient graph*. Any set $A \subseteq E(G)$ such that $\upsilon(G, A) = \xi(G)$ will be called a *Nebeský set*. Furthermore, if A is minimal, then it will be called a *minimal Nebeský set*.

Lemma 2.1([5]) *Let G be a deficient graph and A a minimal Nebeský set of G. Then*

(*a*) *$b(G-A) = c(G-A) \geq 2$. More, if G is a simple graph then every component of $G-A$ contains at least three vertices;*

(*b*) *the end vertices of every edge in A belong to distinct components of $G-A$;*

(*c*) *any two components of $G-A$ are joined by at most one edge of A;*

(*d*) *$\xi(G) = 2c(G-A) - |A| - 1$.*

With the support of Lemma 2.1, we are able to construct a new graph based on the choice of A. Let G be a deficient graph and A a minimal Nebeský set of G. G_A is called a *testable graph of* G if $V(G_A)$ is the set of components of $G-A$ and two vertices in G_A are adjacent if and only if they are joined in G by an edge of A. We shall refer the vertices of G_A to as the *nodes* of G_A, and u_A, υ_A,...are typical notation for the nodes.

Lemma 2.2 *Let G be a deficient graph and A a minimal Nebeský set of G. Then*

$$\xi(G) = 2p(G_A) - q(G_A) - 1,$$

where $p(G_A)$ and $q(G_A)$ are the numbers of nodes and edges of G_A, respectively.

Proof By the definition of G_A, we know that $p(G_A) = c(G-A)$ and $q(G_A) = |A|$. Applying Lemma 2.1, we find that

$$\xi(G) = 2c(G-A) - |A| - 1 = 2p(G_A) - q(G_A) - 1.$$

□

Lemma 2.3 *If G is triangle-free, there exist a $\omega_A \in V(G_A)$ such that $2 \le |E(\omega_A, A)| \le 3$, where $E(\omega_A, A)$ denotes the set of edges of G_A incident with ω_A.*

Proof Let T_{ω_A} denote the component of $G-A$ which corresponds to ω_A in G_A. By Lemma 2.1 $|V(G_A)| \ge 2$. If for all $\omega_A \in V(G_A)$, there is $|E(\omega_A, A)| \ge 4$, then

$$|A| = \frac{1}{2} \sum_{\omega_A \in V(G_A)} |E(\omega_A, A)| \ge 2|V(G_A)|.$$

Applying Lemma 2.1 and the definition of G_A, $\xi(G) = 2V(G_A) - |A| - 1 \le -1$, a contradiction.

For G is connected, $|E(\omega_A, A)| \ge 1$. If $|E(\omega_A, A)| = 1$, let $E(\omega_A, A) = \{e\}$, $e = fh, f \in V(T_{\omega_A}), h \in V(T_{\sigma_A}), \sigma_A \in V(G_A)$. By Lemma 2.1, $\beta(T_{\omega_A})$ is odd and T_{ω_A} is simple and triangle-free, there exists $f' \in V(T_{\omega_A})$ such that $f' \ne f, ff' \notin E(G)$. Similarly, there exists $h' \in V(T_{\sigma_A})$ such that $h' \ne h, hh' \notin E(G)$. Since e is a bridge, $d_G(f', h') \ge 5$, a contradiction.

So we get that $2 \le |E(\omega_A, A)| \le 3$. □

Theorem *Let G be a simple, triangle-free graph of diameter 4, then $\xi(G) \le 2$, i.e., the maximum genus of G, $\gamma_M(G) \ge \frac{1}{2}\beta(G) - 1$.*

Proof Let Π=$\{H|H$ is a simple graph of diameter 4 and does not contain a spanning subgraph K_3 with $\xi(G) > 2$ $\}$. We claim that Π is an empty set. Suppose it is not true, let $G \in \Pi$ be with minimum order. Clearly, G is a deficient graph. Now let A be a minimal Nebeský set. Applying Lemma 2.1(a), each component of $G - A$ has odd Betti number. Thus, each component of $G - A$ must be a quadrangle. Otherwise, there exists a graph $|V(G')| < |V(G)|$. Now let T_{x_A} denote the component of $G - A$ which corresponds to x_A in G_A for each node $x_A \in V(G_A)$.

By Lemma 2.3, choose $z_A \in V(G_A)$ with $2 \le |E(z_A, A)| \le 3$, and define $D_0 = \{z_A\}, D_1 = N(z_A)$ and $D_2 = V(G_A) - N(z_A)$. We call $x \in V(G)$ a distance k vertex, if min $\{d(x,z)|z \in V(T_{z_A})\} = k$ and denote $E(D_i, D_j) = \{x_A y_A \in E(G_A)|x_A \in D_i$ and $y_A \in D_j\}$, where $0 \le i, j \le 2$ (Note that the order of x_A and y_A is important throughout of the proof). We also need the following definitions.

A_1=$\{x_A y_A \in E(D_2, D_1)|$ there exists a distance 1 vertex of T_{y_A} adjacent to a distance 2 vertex of T_{x_A}, or a distance 2 vertex of T_{y_A} adjacent to a distance 3 vertex of T_{x_A} and a distance 1 vertex of T_{ω_A} for some $\omega_A \in D_1 - \{y_A\}\}$.

A_2=$\{x_A y_A \in E(D_2, D_2)|$ x_A is not incident with any edge of A_1 and y_A is incident with one edge of A_1 and T_{y_A} contains a vertex both adjacent to a vertex of T_{x_A} and a vertex of T_{u_A} for some $u_A \in D_1\} \cup \{x_A y_A \in E(D_2, D_2)|$ x_A is not incident with any edge of A_1 and y_A is incident with at least two edges of $A_1\}$.

A_3=$\{x_A y_A \in E(D_1, D_1)|$ there exists a distance 2 vertex of T_{x_A} adjacent to a distance 1 vertex of $T_{y_A}\}$.

Now, according to these edge subsets $A_1 - A_3$ of $E(G_A)$, we define a directed graph $\overrightarrow{G_A}$ based on G_A:

(*i*) $V(\overrightarrow{G_A}) = V(G_A)$;

(*ii*) if $x_A y_A \in E' = (\bigcup_{i=1}^{3} A_i) \bigcup (D_1, D_0)$, then join two arcs from y_A to x_A;

(iii) if $x_Ay_A \in E(G_A) - E^{'}$, then let (x_A, y_A) and (y_A, x_A) be arcs of $\overrightarrow{G_A}$.

By this definition, it is easy to see that

$$\sum_{x_A \in V(G_A)} deg(x_A) = \sum_{x_A \in V(\overrightarrow{G_A})} deg^{-}(x_A),$$

where $deg^{-}(x_A)$ denotes the in-degree of x_A in $\overrightarrow{G_A}$. Therefore, the in-degree sum of $\overrightarrow{G_A}$ gives $2q(G_A)$.

Now, we count the in-degree sum of $\overrightarrow{G_A}$. Let x_A be an arbitrary node in $V(\overrightarrow{G_A})$.

(1) $x_A \in D_0$. Then $deg^{-}(x_A) = 0$ clearly.

(2) $x_A \in D_2$. The situation is divided into the discussions (i)-(iv) following.

(i) x_A is not incident with edges of A_1, but incident with edges of A_2.

Case 1 x_A is incident with at least two edges of A_2, then $deg^{-}(x_A) \geq 4$.

Case 2 x_A is incident with one edge e of A_2. Let x_1y_1 be an edge of $E(G)$ which corresponds to the edge e. Accordingly, T_{z_A} is a quadrangle and $2 \leq |E(z_A, A)| \leq 3$. Then there exist $z_1 \in V(T_{z_A})$ and $deg(z_1) = 2$. We know that $d(x_1, z_1) = 4$ in G. Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$. In T_{x_A}, x_2 must be incident with an edge of $E(G_A) - E^{'}$ such that $d(x_2, z_1) \leq 4$(in fact $d(x_2, z_1) = 4$). Similar discussion can be done done for vertices x_3 and x_4. So $deg^{-}(x_A) \geq 4$.

(ii) x_A is not incident with edges of $A_1 \bigcup A_2$.

Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$. In T_{x_A}, x_1 must be incident with an edge of $E(G_A) - E^{'}$ such that $d(x_1, z_1) \leq 4$(in fact $d(x_2, z_1) = 4$). Similar discussion can be done done for vertices x_2,x_3 and x_4. So $deg^{-}(x_A) \geq 4$.

(iii) x_A is incident with edges of A_1, but not incident with edges of A_2.

Case 1 x_A is incident with at least two edges of A_1, then $deg^{-}(x_A) \geq 4$.

Case 2 x_A is incident with one edge e of A_1. Let x_1y_1 be an edge of $E(G)$ which corresponds to the edge e. Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$ and $d(x_1, z_1) \geq 3$. In T_{x_A}, it supposes that x_3 is not incident with x_1, then x_3 must be incident an edge of $E(G_A)$. Let this edge be $e^{'}$. Then $e^{'} \in E(G_A) - E^{'}$, and $e^{'}$ contributes one de-agree. So $deg^{-}(x_A) \geq 3$(in fact, when $deg^{-}(x_A) = 3$, $e^{'} \in E(D_2, D_1)$).

(iv) x_A is incident with edges of A_1 and A_2.

Case 1 x_A is incident with at least two edges of A_1, then $deg^{-}(x_A) \geq 4$.

Case 2 x_A is incident with one edge e of A_1. Let x_1y_1 be an edge of $E(G)$ which corresponds to the edge e. Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$ and $d(x_1, z_1) \geq 3$. In T_{x_A}, it supposes that x_3 is not incident with x_1, then x_3 must be incident with an edge of $E(G_A)$. Let this edge be $e^{'}$. Then $e^{'} \in A_2$ or $e^{'} \in E(G_A) - E^{'}$. In the former, $e^{'}$ must contributes two de-agree for x_A. In the latter, $e^{'}$ contributes one de-agree. So $deg^{-}(x_A) \geq 3$(in fact,when $deg^{-}(x_A) = 3$, $e^{'} \in E(D_2, D_1)$).

Hence, for $x_A \in D_2$, $deg^{-}(x_A) \geq 3$.

Let M=$\{x_A \in D_2 | deg^-(x_A) = 3\}$. We get that

$$\sum_{x_A \in D_2} deg^-(x_A) \geq 4|D_2| - |M|.$$

(3) $x_A \in D_1$.

By the definition of $\overrightarrow{G_A}$, the edge connects D_0 and D_1 contributes two de-agree for x_A. Let x_1y_1 be an edge of $E(G)$ corresponds to this edge($y_1 \in E(T_{z_A})$).

Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$. In T_{x_A}, it supposes that x_3 is not incident with x_1. In T_{z_A}, there exists $z_2 \in V(T_{z_A})$ so that if $d(x_3, z_2) \leq 4$. If x_3 does not connect z_2 though x_2 or x_4, x_3 must be incident with one edge of $E(G_A)$. Let that edge be e. Then $e \in E(G_A) - E^{'}$ and $deg(x_A) \geq 3$. If x_3 connects z_2 though x_2 or x_4, x_2 or x_4 is incident with one edge of $E(G_A)$. Let that edge be e. Then $e \in E(G_A) - E^{'}$ or $e \in A_3$, and e contributes at least one de-agree. So $deg(x_A) \geq 3$.

Hence, for all $x_A \in D_1$,

$$\sum_{x_A \in D_1} deg^-(x_A) \geq 3|D_1| + |M|.$$

Now by discussions (1) and (2), we get that

$$\begin{aligned} 2q(G_A) &= \sum_{x_A \in V(\overrightarrow{G}_A)} deg^-(x_A) \\ &\geq 4|D_2| - |M| + 3|D_1| + |M| \\ &= 4p(G_A) - |D_1| - 4 \\ &\geq 4P(G_A) - 7. \end{aligned}$$

Applying Lemma 2.2 again, we get that $\xi(G) = 2p(G_A) - 1 - q(G_A) \leq 2$, also a contradiction. This completes the proof. □

To see that the upper bound presented in our theorem is best possible, let us consider the following family of infinite graphs, as depicts in Fig. 1. There are even paths with length 2 from m to n. Thus, this graph is triangle-free with diameter 4. It is not difficult to check that its Betti deficiency are equal to 2.

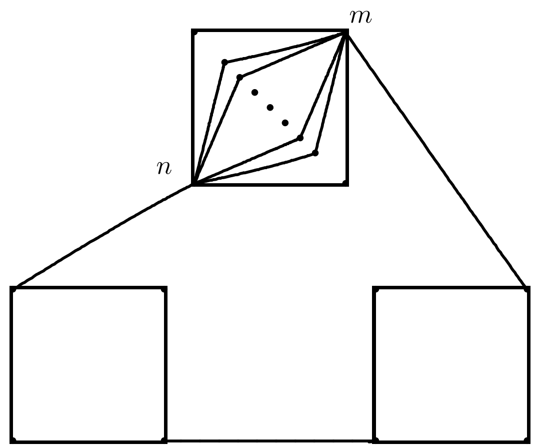

Fig.1

References

[1] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[2] L.Nebeský, A new characterization of the maximum genus of a graph,*Czechosolvak Math.J.*, 31(106)(1981),604-613.

[3] Skoviera M., The maximum genus of graphs of diameter two, *Discrete Mathematics,* 1991,87:175-180.

[4] Skoviera M., The decay number and the maximum genus of a graph, *Mathematics Slovaca*, 1992, 42(4):391-406.

[5] Hunglin Fu, Minchu Tsai, The maximum genus of diameter three graphs, *Australasian J Combinatorics,* 1996,14:187- 197.

[6] Yuangqiu Huang, Yanpei Liu, On the Maximum Genus of Graphs with Diameter Four, *Mathematica Scientia,* 2001, 21A(3):349-354.

[7] Hung-Lin Fu, Ming-chun Tsai, The Maximum Genus of a Graph with Given Diameter and Connectivity, *Electronic Notes of Discrete Math.*, Vol.11(2002).

[8] N.H.Xuong, How to determine the maximum genus of a graph, *J.Combin.Theory*(B), 26(1979),217-225.

[9] J.Chen, D.Archdeacon and J.L.Gross, Maximum genus and connectivity, *Discrete Math.*, 149(1996),19-39.

[10] Nordhaus E, Stewart B and White A., On the maximum genus of a graph, *J Combinatorial Theory* B,1971,11:258-267.

[11] M.Knor, J.Širáň, Extremal graphs of diameter two and given maximum degree, embeddable in a fixed surface, *J. Graph Theory*, 24 (1997) No. 1, 1-8.

[12] D. Archdeacon, P.Bonnington, J.Širáň, A Nebesky-type characterization for relative maximum genus, *J. Combinat. Theory* (B), 73 (1998), 77-98.

[13] Kanchi S.P and Chen J, A tight lower bound on the maximum genus of a 2-connected simplicial graph, manuscript(1992).

[14] Chen J,Kanchi S.P,Gross J.L., A tight lower bound on the maximum genus of a simplicial graph, *Discrete Math.*,1996,156:83-102.

[15] J.Širáň, M. Škoviera, Characterization of the maximum genus of a signed graph, *J. Combinat. Theory* (B) 52 (1991), 124-146.

[16] M.C.Tsai, *A study of maximum genus via diameter*, Ph.D.thesis, 1996, National Chiao Tung University, Hsin Chu,Tainwan,R.O.C.

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Long Dominating Cycles in Graphs

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Abstract: Let G be a connected graph of order n, and $NC2(G)$ denote $\min\{|N(u)\cup N(v)| : dist(u,v)=2\}$, where $dist(u,v)$ is the distance between u and v in G. A cycle C in G is called a *dominating cycle*, if $V(G)\backslash V(C)$ is an independent set in G. In this paper, we prove that if G contains a dominating cycle and $\delta\geq 2$, then G contains a dominating cycle of length at least $\min\{n, 2NC2(G)-1\}$ and give a family of graphs showing our result is sharp, which proves a conjecture of R. Shen and F. Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces.

Key words: Dominating cycle, neighborhood union, distance.

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§1. Introduction

All graphs considered in this paper will be finite and simple. We use Bondy & Murty [1] for terminology and notations not defined here.

Let $G=(V,E)$ be a graph of order n and C be a cycle in G. C is called a *dominating cycle*, or briefly a *D-cycle*, if $V(G)\backslash V(C)$ is an independent set in G. For a vertex v in G, the neighborhood of v is denoted by $N(v)$, and the degree of v is denoted by $d(v)$. For two subsets S and T of $V(G)$, we set $N_T(S)=\{v\in T\backslash S : N(v)\cap S\neq\emptyset\}$. We write $N(u,v)$ instead of $N_{V(G)}(\{u,v\})$ for any $u,v\in V(G)$. If F and H are two subgraphs of G, we also write $N_F(H)$ instead of $N_{V(F)}\big(V(H)\big)$. In the case $F=G$, if no ambiguity can arise, we usually omit the subscript G of $N_G(H)$. We denote by $G[S]$ the subgraph of G induced by any subset S of $V(G)$.

For a connected graph G and $u,v\in V(G)$, we define the *distance between u and v* in G, denoted by $dist(u,v)$, as the minimum value of the lengths of all paths joining u and v in G. If G is non-complete, let $NC(G)$ denote $\min\{|N(u,v)| : uv\notin E(G)\}$ and $NC2(G)$ denote $\min\{|N(u,v)| :\ dist(u,v)=2\}$; if G is complete, we set $NC(G)=n-1$ and $NC2(G)=n-1$.

In [2], Broersma and Veldman gave the following result.

Theorem 1([2]) *If G is a 2-connected graph of order n and G contains a D-cycle, then G has a D-cycle of length at least* $\min\{n, 2NC(G)\}$ *unless G is the Petersen graph.*

For given positive integers n_1,n_2 and n_3, let $K(n_1,n_2,n_3)$ denote the set of all graphs

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of order $n_1 + n_2 + n_3$ consisting of three disjoint complete graphs of order n_1, n_2 and n_3, respectively. For any integer $p \geq 3$, let $\mathcal{J}_1^*$ (resp. $\mathcal{J}_2^*$) denote the family of all graphs of order $2p+3$ (resp. $2p+4$) which can be obtained from a graph H in $K(3,p,p)$ (resp. $K(3,p,p+1)$) by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of H. Let $\mathcal{J}_1 = \{G : G$ is a spanning subgraph of some graph in $\mathcal{J}_1^*\}$ and $\mathcal{J}_2 = \{G : G$ is a spanning subgraph of some graph in $\mathcal{J}_2^*\}$. In [5], Tian and Zhang got the following result.

Theorem 2([5]) *If G is a 2-connected graph of order n such that every longest cycle in G is a D-cycle, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G)\}$ *unless G is the Petersen graph or $G \in \mathcal{J}_1 \cup \mathcal{J}_2$.*

In [4], Shen and Tian weakened the conditions of Theorem 2 and obtained the following theorem.

Theorem 3([4]) *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - 3\}$.

Theorem 4([6]) *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - 2\}$.

In [4], Shen and Tian believed the followings are true.

Conjecture 1 *If G satisfies the conditions of Theorem 3, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - \epsilon(n)\}$, *where $\epsilon(n) = 1$ if n is even, and $\epsilon(n) = 2$ if n is odd.*

Conjecture 2 *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G)\}$ *unless G is one of the exceptional graphs listed in Theorem 2. And the complete bipartite graphs $K_{m,m+q}$ $(q \geq 1)$ show that the bound $2NC2(G)$ is sharp.*

In this paper, we prove the following result, which solves Conjecture 1 due to Shen and Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces (see [3] for details).

Theorem 5 *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - 1\}$ *unless $G \in \mathcal{J}_1$.*

Remark The Petersen graph shows that our bound $2NC2(G) - 1$ is sharp.

§2. Proof of Theorem 5

In order to prove Theorem 5, we introduce some additional notations.

Let C be a cycle in G. We denote by $\overrightarrow{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to

denote its predecessor. We write $u^{+2} := (u^+)^+$ and $u^{-2} := (u^-)^-$, etc. If $A \subseteq V(C)$, then $A^+ = \{v^+ : v \in A\}$ and $A^- = \{v^- : v \in A\}$. For any subset S of $V(G)$, we write $N^+(S)$ and $N^-(S)$ instead of $(N(S))^+$ and $(N(S))^-$,respectively.

Let G be a graph satisfying the conditions of Theorem 4, i.e. G contains a D-cycle and $\delta \geq 2$. Throughout, we suppose that

——G is non-hamiltonian and C is a longest D-cycle in G,

——$|V(C)| \leq 2NC2(G) - 2$,

——$R = G \backslash V(C)$ and $x \in R$, such that $d(x)$ is as large as possible.

First of all, we prove some claims.

By the maximality of C and the definition of D-cycle, we have

Claim 1 $N(x) \subseteq V(C)$.

Claim 2 $N(x) \cap N^+(x) = N(x) \cap N^-(x) = \emptyset$.

Let $v_1, v_2, \ldots, v_k$ be the vertices of $N(x)$, in cyclic order around $\overrightarrow{C}$. Then $k \geq 2$ since $\delta \geq 2$. For any $i \in \{1, 2, \ldots, k\}$, we have $v_i^+ \neq v_{i+1}$ (indices taken modulo k) by Claim 2. Let $u_i = v_i^+$, $w_i = v_{i+1}^-$ (indices taken modulo k), $T_i = u_i \overrightarrow{C} w_i$, $t_i = |T_i|$.

Claim 3 $N_R(y_1) \cap N_R(y_2) = \emptyset$, if y_1, $y_2 \in N^+(x)$ or y_1, $y_2 \in N^-(x)$.In particular, $N^+(x) \cap N(u_i) = N^-(x) \cap N(w_i) = \emptyset$.

For any $i, j \in \{1, 2, \ldots, k\} (i \neq j)$, we also have the following Claims.

Claim 4 Each of the followings does not hold :

(1) There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^-]$, $(z \in v_{j+1} \overrightarrow{C} v_i)$ of length at most two that are internally disjoint from C and each other ;

(2) There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^+]$ $(z \in v_{j+1} \overrightarrow{C} v_i)$ of length at two that are internally disjoint from C and each other ;

(3) There are two paths $P_1[u_i, z]$ and $P_2[u_j, z^+]$ $(z \in u_j^+ \overrightarrow{C} v_i)$ of length at most two that are internally disjoint from C and each other, and similarly for $P_1[u_i, z]$ and $P_2[u_j, z^-]$ $(z \in u_i^+ \overrightarrow{C} v_j)$.

Claim 5 For any $v \in V(G)$, we have $d_R(v) \leq 1$.

If not, then by Claim 1, there exists a vertex, say v, in C such that $d_R(v) > 1$. Let $x_1, x_2 \in N_R(v)$, then $|N(x_1, x_2)| \geq NC2(G)$.

First, we prove that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$. Otherwise, let y_1, y_2 and y_3 be three distinct vertices in $N(x_1, x_2) \cap N^+(x_1, x_2)$. By Claim 2, we know $y_i \in N(x_1) \cap N^+(x_2)$ or $y_i \in N(x_2) \cap N^+(x_1)$ for any $i \in \{1, 2, 3\}$. Thus, there must exist i and j $(i \neq j, i, j \in \{1, 2, 3\})$ such that $y_i, y_j \in N(x_1) \cap N^+(x_2)$ or $y_i, y_j \in N(x_2) \cap N^+(x_1)$. In either case, it contradicts Claim 3. So we have that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$.

Now we have

$$\begin{aligned} |V(C)| &\geq |N(x_1,x_2)\cup N^+(x_1,x_2)| \\ &\geq 2|N(x_1,x_2)|-2 \\ &\geq 2NC2(G)-2, \end{aligned}$$

so $V(C)=N(x_1,x_2)\cup N^+(x_1,x_2)$ by assumption on $|V(C)|$, and in particular, $N(x_1,x_2)\cap N^+(x_1,x_2)=\{y_1,y_2\}$.Therefore $y_1\in N(x_1)\cap N^+(x_2)$ and $y_2\in N^+(x_1)\cap N(x_2)$.

Now, we prove that $d_R(v^+)\leq 1, d_R(v^-)\leq 1$. If not,suppose $d_R(v^-)>1$, let $z_1,z_2\in N_R(v^-)$, by Claim 1 and $V(C)=N(x_1,x_2)\cup N^+(x_1,x_2), N(z_1,z_2)\subseteq N^+(x_1,x_2)$, so we have x_1 (or x_2) $\in N(v^{-2})$. Using a similar argument as above, we have z_1 (or z_2) $\in N(v^{-3})$, which contradicts Claim 3. Thus, we have $d_R(v^-)\leq 1$; similarly, $d_R(v^+)\leq 1$.

Now, we consider $N(x_2,v^-)\cup N^-(x_1,v^+)$.Since $dist(x_2,v^-)=dist(x_1,v^+)=2$ and $|N(x_2,v^-)|\geq NC2(G), |N^-(x_1,v^+)|=|N(x_1,v^+)|\geq NC2(G)$. We prove that $|N_C(x_2,v^-)\cap N_C^-(x_1,v^+)|\leq 1$. Let $z\in\{N_C(x_2,v^-)\cap N_C^-(x_1,v^+)\}\backslash\{y_1^-\}$.

We consider following cases.

(i) Let $z\in y_1^+\overrightarrow{C}y_2^{-2}$, if $zx_2\in E(G)$ and $x_1z^+\in E(G)$, or $zx_2\in E(G)$ and $v^+z^+\in E(G)$, or $v^-z\in E(G)$ and $x_1z^+\in E(G)$, each case contradicts Claim 3; if $v^-z\in E(G)$ and $v^+z^+\in E(G)$, then $C'=x_1y_2^-\overleftarrow{C}z^+v^+\overrightarrow{C}zv^-\overleftarrow{C}y_2x_2vx_1$ is a D-cycle longer than C,a contradiction.

(ii) Let $z\in y_2^+\overrightarrow{C}y_1^{-2}$, if $x_2z\in E(G)$ and $x_1z^+\in E(G)$, or $x_2z\in E(G)$ and $v^+z^+\in E(G)$, both contradict Claim 3; if $v^-z\in E(G)$ and $x_1z^+\in E(G)$, it contradicts Claim 3; if $v^-x_1\in E(G)$ and $z^+v^+\in E(G)$, then $C'=x_1y_1\overrightarrow{C}v^-z\overleftarrow{C}v^+z^+\overrightarrow{C}y_1^-x_2vx_1$ is a D-cycle longer than C, for $z\in v\overrightarrow{C}y_1^-$; and $C'=x_1y_2^-\overleftarrow{C}v^+z^+\overrightarrow{C}v^-z\overleftarrow{C}y_2x_2vx_1$ is a D-cycle longer than C for $z\in y_2\overrightarrow{C}v^-$.

So, we have $|N_C(x_2,v^-)\cap N_C^-(x_1,v^+)|\leq 1$. Moreover, $y_1,y_2^-\notin N(x_2,v^-)\cup N^-(x_1,v^+)$. Otherwise, if $y_1\in N(v^-)$, then $C'=x_1y_2^-\overleftarrow{C}y_1v^-\overleftarrow{C}y_2x_2y_1^-\overleftarrow{C}vx_1$ is a D-cycle longer than C. By Claim 2, $y_1\notin N(x_2)\cup N^-(x_1,v^+)$, so we have $y_1\notin N(x_2,v^-)\cup N^-(x_1,v^+)$. By Claims 1 and 3 we have $y_2^-\notin N(x_2,v^-)\cup N^-(x_1,v^+)$. Thus, we have

$$\begin{aligned} |V(C)| &\geq |N_C(x_2,v^-)\cup N_C^-(x_1,v^+)|+2 \\ &\geq |N_C(x_2,v^-)|+|N_C^-(x_1,v^+)|-1+2 \\ &= |N(x_2,v^-)\backslash N_R(x_2,v^-)|+|N(x_1,v^+)\backslash N_R(x_1,v^+)|+1 \\ &\geq 2NC2(G)-2+1 \\ &= 2NC2(G)-1, \end{aligned}$$

a contradiction with $|V(C)|\leq 2NC2(G)-2$. So, we have $d_R(v)\leq 1$, for any $v\in V(G)$.

Claim 6 $t_i\geq 2$.

If $t_i=1$ for all of i, then $N_R(u_i)=\emptyset$ for all of i (if not, let $z\in N_R(u_i)$ for some i, by Claim 1 and Claim 5 $N(z)\subseteq V(C)$ and $u_jz\in E(G)$ for some j. then, $z\in N_R(u_i)\cap N_R(u_j)$, a contradiction). Then $N(u_i)\cap N^+(u_i)=\emptyset$(otherwise, $y\in N(u_i)\cap N^+(u_i)$, then $C'=$

$xv_{i+1}\overrightarrow{C}y^-u_iy\overrightarrow{C}v_ix$ is a D-cycle longer than C). Moreover, we have $N(x)\cap N^+(x)=\emptyset$ by Claim 2, $N^+(x)\cap N(u_i)=N^+(u_i)\cap N(x)=\emptyset$ by Claim 3. Hence, $N(x,u_i)\cap N^+(x,u_i)=\emptyset$. So we have

$$|V(C)|\geq|N(x,u_i)\cup N^+(x,u_i)|\geq 2|N(x,u_i)|\geq 2NC2(G),$$

a contradiction. So we may assume $t_i=1$ for some i, without loss of generality, suppose $t_1=1$ and $N_R(w_k)\neq\emptyset$. Let $y\in N_R(w_k)$, choose $y_1\in N(y)$ such that $N(y)\cap(y_1^+\overrightarrow{C}w_k^-)=\emptyset$. Using a similar argument as above and $d_R(u_1)\leq 1$, by Claim 5, we have

$$|V(C)|=|N_C(x,u_1)\cup N_C^+(x,u_1)|\geq 2NC2(G)-2.$$

So $V(C)=N_C(x,u_1)\cup N_C^+(x,u_1)$. Similarly, we know that $V(C)=N_C(x,u_1)\cup N_C^-(x,u_1)$. Moreover, $u_1w_k^-\in E(G)$. If $|y_1^+\overrightarrow{C}w_k^-|=1$, then $C'=xv_2\overrightarrow{C}y_1yw_kw_k^-u_1v_1x$ is a D–cycle longer than C, a contradiction. So we may assume that $|y_1^+\overrightarrow{C}w_k^-|\geq 2$.

Now, we consider $N_C(y,y_1^+)\cup N_C^-(x,u_1)$. Since $dist(y,y_1^+)=dist(x,u_1)=2$, $|N(y,y_1^+|\geq NC2(G), |N^-(x,u_1)|=|N(x,u_1)|\geq NC2(G)$. Moreover, we have $v_1,v_2\notin N_C(y,y_1^+)\cup N_C^-(x,u_1)$ and $N_C(y,y_1^+)\cap N_C^-(x,u_1)\subseteq\{w_k\}$. In fact, $v_1\notin N(y,y_1^+)$ by Claims 3 and 5, if $v_1\in N^-(x,u_1)$, then $v_1^+x\in E(G)$ or $v_1^+u_1\in E(G)$, which contradicts to Claims 2 and 3. So $v_1\notin N_C(y,y_1^+)\cup N_C^-(x,u_1)$;if $v_2\in N_C(y,y_1^+)$, then $v_2y^+\in E(G)$ by Claim 5, which contradicts to Claim 4. If $v_2\in N_C^-(x,u_1)$ then $v_2^+\in N(x,u_1)$, which contradicts to Claims 2 and 3. So $v_2\notin N_C(y,y_1^+)\cup N_C^-(x,u_1)$. Suppose $z\in N_C(y,y_1^+)\cap N_C^-(x,u_1)\backslash\{w_k\}$. Now, we consider the following cases.

(i) $z\in v_2\overrightarrow{C}y_1^-$. If $yz\in E(G)$ and $xz^+\in E(G)$, then, it contradicts to Claim 3. Put

$$C'=\begin{cases} yz\overleftarrow{C}v_2xv_1u_1z^+\overleftarrow{C}w_ky & \text{if } yz\in E(G)\text{and}u_1z^+\in E(G);\\ xz^+\overrightarrow{C}y_1yw_k\overleftarrow{C}y_1^+z\overleftarrow{C}v_1x & \text{if } y_1^+z\in E(G)\text{ and}xz^+\in E(G);\\ xv_2\overrightarrow{C}zy_1^+\overrightarrow{C}w_kyy_1\overleftarrow{C}z^+u_1v_1x & \text{if } y_1^+z\in E(G)\text{ and } u_1z^+\in E(G).\end{cases}$$

(ii) $z\in y_1\overrightarrow{C}w_k^-$, then $z\in N(y_1^+)$ since $N(y)\cap(y_1^+\overrightarrow{C}w_k^-)=\emptyset$. Let $zy_1^+\in E(G)$ and $z^+\in N_C(x,u_1)$. Since $V(C)=N_C(x,u_1)\cup N_C^-(x,u_1)$, So $y_1^+\in N_C(x,u_1)\cup N_C^-(x,u_1)$. If $u_1y_1^+\in E(G)$ then $C'=xv_2\overrightarrow{C}y_1yw_k\overleftarrow{C}y_1^+u_1v_1x$ is a D–cycle longer than C , a contradiction; if $xy_1^+\in E(G)$, then it contradicts with Claim 3. Then, $y_1^+\in N^-(x,u_1)$. If $xz^+\in E(G)$ and $y_1^{+2}x\in E(G)$, then it contradicts to Claim 3; Put

$$C'=\begin{cases} xy^{+2}\overrightarrow{C}zy_1^+\overleftarrow{C}u_1z^+\overleftarrow{C}v_1x & \text{if } y_1^{+2}x\in E(G)\text{ and } u_1z^+\in E(G);\\ xv_2\overrightarrow{C}y_1^+z\overleftarrow{C}y_1^{+2}u_1\overleftarrow{C}z^+x & \text{if } y_1^{+2}u_1\in E(G)\text{ and } xz^+\in E(G);\\ xv_2\overrightarrow{C}y_1^+z\overleftarrow{C}y_1^{+2}u_1z^+\overleftarrow{C}v_1x & \text{if } y_1^{+2}u_1\in E(G)\text{ and } u_1z^+\in E(G).\end{cases}$$

In any cases, C' is a D–cycle longer than C, a contradiction. Therefore, $v_1,v_2\notin N_C(y,y_1^+)\cup N_C^-(x,u_1), N_C(y,y_1^+)\cap N_C^-(x,u_1)\subseteq\{w_k\}$. Hence, we have

$$\begin{aligned} |V(C)| &\geq |N_C(y,y_1^+)\cup N_C^-(x_1,u_1)|+2\\ &\geq |N_C(y,y_1^+)|+|N_C^-(x_1,u_1)|-1+2\\ &= |N(y,y_1^+)\backslash N_R(y,y_1^+)|+|N(x_1,u_1)\backslash N_R(x_1,u_1)|+1\\ &\geq 2NC2(G)-2+1\\ &= 2NC2(G)-1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$.

Claim 7 If $\bigcup_{i=1}^{k} N_R(y_i) \neq \emptyset$, then $N_R(y_i) \neq \emptyset$ for all $i \in \{1, 2, \ldots, k\}$, where $y_i = u_i$ (w_i, respectively).

If not, without loss of generality, we assume that $N_R(u_1) \neq \emptyset$ and $N_R(u_k) = \emptyset$. Suppose $x_1 \in N_R(u_1)$ and $y \in N(x_1)$ ($y \neq u_1$). Then $dist(x_1, y^+) = dist(x_1, y^-) = 2$ and $|N(x_1, y^+)| \geq NC2(G)$, $|N(x_1, y^-)| \geq NC2(G)$.

Case 1 $N(x_1) \cap (u_1^+ \overrightarrow{C} v_k) = \emptyset$.

If not, we may choose $y, y \in N(x_1) \cap (u_1^+ \overrightarrow{C} v_k)$, such that $N(x_1) \cap (u_1^+ \overrightarrow{C} y^-) = \emptyset$. We define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^- & \text{if } v \in u_k \overrightarrow{C} y^-; \\ v^+ & \text{if } v \in y \overrightarrow{C} w_{k-1}; \\ y^- & \text{if } v = v_k. \end{cases}$$

Then $|f(N_C(x, u_k))| = |N_C(x, u_k)| = |N(x, u_k)| \geq NC2(G)$ by Claim 1 and the assumption $N_R(u_k) = \emptyset$. Moreover, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. In fact, suppose that $z \in f(N_C(x, u_k)) \cap N(x_1, y^-) \backslash \{w_k, u_1\}$. Obviously, $z \neq v_1, y^-$ by Claims 2 and 4. Now we consider the following cases.

(i) If $z \in u_k \overrightarrow{C} w_k^-$, then $z \in N_C^-(u_k)$ since $N(x) \cap (u_k \overrightarrow{C} w_k) = \emptyset$. Put

$$C' = \begin{cases} u_k z^+ \overrightarrow{C} v_1 x v_k \overleftarrow{C} u_1 x_1 z \overleftarrow{C} u_k & \text{if } x_1 z \in E(G); \\ u_k z^+ \overrightarrow{C} v_1 x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- z \overleftarrow{C} u_k & \text{if } y^- z \in E(G). \end{cases}$$

(ii) If $z \in u_1^+ \overrightarrow{C} y^{-2}$, then $zy^- \in E(G)$ since $N(x_1) \cap (u_1^+ \overrightarrow{C} y^-) = \emptyset$. Put

$$C' = \begin{cases} u_1 \overrightarrow{C} z y^- \overleftarrow{C} z^+ x v_1 \overleftarrow{C} y x_1 u_1 & \text{if } xz^+ \in E(G); \\ u_1 \overrightarrow{C} z y^- \overleftarrow{C} z^+ u_k \overrightarrow{C} v_1 x v_k \overleftarrow{C} y x_1 u_1 & \text{if } u_k z^+ \in E(G). \end{cases}$$

(iii) If $z \in y^+ \overrightarrow{C} v_k$, we put

$$C' = \begin{cases} u_1 \overrightarrow{C} z^- x v_1 \overleftarrow{C} z x_1 u_1 & \text{if} xz^- \in E(G) \text{ and } x_1 z \in E(G); \\ u_1 \overrightarrow{C} y^- z \overrightarrow{C} v_1 x z^- \overleftarrow{C} y x_1 u_1 & \text{if } xz^- \in E(G) \text{ and } y^- z \in E(G); \\ u_1 \overrightarrow{C} z^- u_k \overrightarrow{C} v_1 x v_k \overleftarrow{C} z x_1 u_1 & \text{if } u_k z^- \in E(G) \text{ and } x_1 z \in E(G); \\ u_1 \overrightarrow{C} y^- z \overrightarrow{C} v_k x v_1 \overleftarrow{C} u_k z^- \overleftarrow{C} y x_1 u_1 & \text{if } u_k z^- \in E(G) \text{ and } y^- z \in E(G). \end{cases}$$

In any cases, C' is a D-cycle longer than C, a contradiction. Therefore, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. By Claims 2 and 4, we have $u_1 \notin N(x, u_k)$ and $v_1 \notin N(x_1, y^-)$. Then $v_1 \notin f(N_C(x, u_k)) \cup N(x_1, y^-)$. Hence, by Claim 6 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 1 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 2 + 1 \\ &\geq 2NC2(G) - 2. \end{aligned}$$

So, we have $V(C) = N_C(x_1, y^-) \cup f(N_C(x, u_k)) \cup \{v_1\}, N_C(x_1, y^-) \cap f(N_C(x, u_k)) = \{w_k, u_1\}$. Hence, $y^- w_k \in E(G)$ and $u_k u_1^+ \in E(G)$ since $t_i \geq 2$.

Now, we prove that $N_R(y^-) = \emptyset$. If not, there exist $y_1 \in N_R(y^-), z \in N_C(y_1)\,(z \neq y^-)$ by Claim 1 and $\delta \geq 2$.

Subcase 1 $N(y_1) \cap (u_1 \overrightarrow{C} y^{-2}) = \emptyset$.

If not, we choose $z \in N(y_1)$, such that $N(y_1) \cap (z^+ \overrightarrow{C} y^{-2}) = \emptyset$. Therefore we can define a mapping f_1 on $V(C)$ as follows:

$$f_1(v) = \begin{cases} v^- & \text{if } v \in u_k^+ \overrightarrow{C} z^+; \\ v^+ & \text{if } v \in z^{+2} \overrightarrow{C} w_{k-1}; \\ z^{+2} & \text{if } v = v_k; \\ z^+ & \text{if } v = u_k. \end{cases}$$

Using an argument as above , we have $|f_1(N_C(x, u_k)| \geq NC2(G)$. Moreover, we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$ and $N_C(y_1, z^+) \cap f_1(N_C(x, u_k)) \subseteq \{z^{+2}, y^-, w_k\}$. Clearly, $z^+ \notin N_C(y_1, z^+)$. If $z^+ \in f_1(N_C(x, u_k))$, then, $u_k \in N_C(x, u_k)$, a contradiction. $y_1 v_1 \notin E(G)$ by Claim 5. If $v_1 z^+ \in E(G)$, since $y, z^+ \in N^+(y_1)$, the two paths $y x_1 u_1$ and $z^+ v_1$ contradict with Claim 4; By Claims 2 and 4 , we have $y \notin N(y_1, z^+)$, if $y \in f_1(N_C(x, u_k))$ then $y^- \in N_C(x, u_k)$, by Claim 3 $y^- \notin N(x)$, so $y^- \in N(u_k)$, then $C' = x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- u_k \overrightarrow{C} v_1 x$ is a D –cycle longer than C, a contradiction. So we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$. Suppose $s \in N_C(y_1, z^+) \cap f_1(N_C(x, u_k)) \backslash \{z^{+2}, y^-, w_k\}$.

Now, we consider the following cases.

(i) $s \in y^+ \overrightarrow{C} v_k$. If $y_1 s \in E(G)$ and $x s^- \in E(G)$ then it contradicts with Claim 4. We put

$$C' = \begin{cases} x v_k \overleftarrow{C} s y_1 y^- \overleftarrow{C} u_1 x_1 y \overleftarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^- \in E(G); \\ x s^- \overleftarrow{C} y x_1 u_1 \overrightarrow{C} z y_1 y^- \overleftarrow{C} z^+ s \overrightarrow{C} v_1 x & \text{if } z^+ s, x s^- \in E(G); \\ x v_k \overleftarrow{C} s z^+ \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_1 x_1 y \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^+ s, u_k s^- \in E(G). \end{cases}$$

(ii) $s \in u_k \overrightarrow{C} w_{k-1}$. We have $s \in N^-(u_k)$ since $N(x) \cap (u_k \overrightarrow{C} w_k) = \emptyset$.Put

$$C' = \begin{cases} x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- y_1 s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^+ \in E(G); \\ x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} z y_1 y^- \overleftarrow{C} z^+ s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } z^+ s, u_k s^+ \in E(G); \end{cases}$$

(iii) $s \in u_1 \overrightarrow{C} y^{-2}$. If $y_1 s, x s^+ \in E(G)$ then contradicts to Claim 4. If $y_1 s, u_k s^+ \in E(G)$, then

$$C' = x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} s y_1 y^- \overleftarrow{C} s^+ u_k \overrightarrow{C} v_1 x$$

is a D–cycle longer than C, a contradiction. If $s \in z^+ \overrightarrow{C} y^-$, we put

$$C' = \begin{cases} x s^- \overleftarrow{C} z^+ s \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_1 x_1 y \overrightarrow{C} v_1 x & \text{if } z^+ s, s^- x \in E(G); \\ x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} z y_1 y^- \overleftarrow{C} s z^+ \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^+ s, s^- u_k \in E(G). \end{cases}$$

If $s \in u_1\overrightarrow{C}z$, we put

$$C' = \begin{cases} xs^+\overrightarrow{C}zy_1y^-\overleftarrow{C}z^+s\overleftarrow{C}u_1x_1y\overrightarrow{C}v_1x & \text{if } z^+s, xs^+ \in E(G); \\ xv_k\overleftarrow{C}yx_1u_1\overrightarrow{C}sz^+\overrightarrow{C}y^-y_1z\overleftarrow{C}s^+u_k\overrightarrow{C}v_1x & if\ z^+s, u_ks^+ \in E(G). \end{cases}$$

In any cases, C' is a D–cycle longer than C, a contradiction. Hence, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f_1(N_C(x,u_k)) \cup N_C(y_1,z^+)| + 3 \\ &\geq |f_1(N_C(x,u_k))| + |N_C(y_1,z^+)| - 3 + 3 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So $N(y_1) \cap (u_1\overrightarrow{C}y^{-2}) = \emptyset$,

Subcase 2 $N(y_1) \cap (y\overrightarrow{C}v_k) = \emptyset$.

If not, we may choose $z \in N(y_1) \cap (y\overrightarrow{C}v_k)$, such that $N(y_1) \cap (y\overrightarrow{C}z^-) = \emptyset$. Therefore, we can define a mapping f_2 on $V(C)$ as follows:

$$f_2(v) = \begin{cases} v^+ & \text{if } v \in u_1\overrightarrow{C}y^{-2} \cup z^-\overrightarrow{C}w_{k-1}; \\ v^- & \text{if } v \in y^+\overrightarrow{C}z^{-2} \cup u_k^+\overrightarrow{C}v_1; \\ z^- & \text{if } v = v_k; \\ v_1 & \text{if } v = u_k; \\ z^{-2} & \text{if } v = y; \\ u_1 & \text{if } v = y^- \end{cases}$$

Using a similar argument as above , we have $|f_2(N_C(x,u_k))| \geq NC2(G)$. We consider $N_C(y_1,z^-) \cup f_2(N_C(x,u_k))$, then $v_1, u_1^+ \notin N_C(y_1,z^-) \cup f_2(N_C(x,u_k))$, and $N_C(y_1,z^-) \cap f_2(N_C(x,u_k)) \subseteq \{y^-, w_k\}$. In fact, $v_1 \notin N(y_1,z^-)$ by Claims 4, 5 ; if $v_1 \in f_2(N(x,u_k))$ then $u_k \in N(x,u_k)$, a contradiction; if $u_1^+ \in N(z^-)$, then the paths yx_1u_1 and $z^-u_1^+$ contradict with Claim 5; if $u_1^+ \in f_2(N_C(x,u_k))$, then $u_1 \in N(x,u_k)$, a contradiction. So we have $v_1, u_1^+, \notin N_C(y_1,z^-) \cup f_2(N_C(x,u_k))$. For $s \in N_C(y_1,z^-) \cap f_2(N_C(x,u_k)) \backslash \{y^-, w_k\}$, we consider the following cases.

(i) If $s \in u_1\overrightarrow{C}y$. We have $s \in N(z^-)$ since $N(y_1) \cap (u_1\overrightarrow{C}y^{-2}) = \emptyset$. Put

$$C' = \begin{cases} xs^-\overleftarrow{C}u_1x_1y\overrightarrow{C}z^-s\overrightarrow{C}y^-y_1z\overrightarrow{C}v_1x & \text{if } s^-x \in E(G); \\ xv_k\overleftarrow{C}zy_1y^-\overleftarrow{C}sz\overleftarrow{C}yx_1u_1\overrightarrow{C}s^-u_k\overrightarrow{C}v_1x & \text{if } s^-u_k \in E(G). \end{cases}$$

(ii) If $s \in u_k\overrightarrow{C}v_1$, then $s^+ \in N(u_k)$ since $N(x) \cap (u_k\overrightarrow{C}w_k) = \emptyset$. Put

$$C' = \begin{cases} xv_k\overleftarrow{C}zy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}z^-s\overleftarrow{C}u_ks^+\overrightarrow{C}v_1x & \text{if } z^-s \in E(G); \\ xv_k\overleftarrow{C}yx_1u_1\overrightarrow{C}y^-y_1s\overleftarrow{C}u_ks^+\overrightarrow{C}v_1x & \text{if } y_1s \in E(G). \end{cases}$$

(iii) If $s \in y\overrightarrow{C}z^{-2}$, then we have $s \in N(z^-)$ since $N(y_1) \cap (y\overrightarrow{C}z^{-2}) = \emptyset$. Put

$$C' = \begin{cases} x_1y\overrightarrow{C}sz^-\overleftarrow{C}s^+xv_1\overleftarrow{C}zy_1y^-\overleftarrow{C}u_1x_1 & \text{if } xs^+ \in E(G); \\ xv_k\overleftarrow{C}zy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}sz^-s^+u_k\overleftarrow{C}v_1x & \text{if } u_ks^+ \in E(G). \end{cases}$$

(iv) If $s \in z^-\overrightarrow{C}v_k$. If $y_1s, xs^- \in E(G)$ then it contradicts to Claim 4. We put

$$C' = \begin{cases} xv_k\overleftarrow{C}sy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}s^-u_k\overrightarrow{C}v_1x & \text{if } y_1s, u_ks^- \in E(G); \\ xs^-\overleftarrow{C}zy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}z^-s\overrightarrow{C}v_1x & \text{if } z^-s, s^-x \in E(G); \\ xv_k\overleftarrow{C}sz^-\overleftarrow{C}yx_1u_1\overrightarrow{C}y^-y_1z\overrightarrow{C}s^-u_k\overrightarrow{C}v_1x & \text{if } z^-s, s^-u_k \in E(G). \end{cases}$$

In any cases, C' is a D-cycle longer than C, a contradiction. Therefore, we have $v_1, u_1^+, \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k))$, and $N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. So

$$\begin{aligned} |V(C)| &\geq |N_C(y_1, z^-) \cup f_2(N_C(x, u_k))| + 2 \\ &\geq |N_C(y_1, z^-)| + |N_C(x, u_k)| - 2 + 2 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$. Hence, $N(y_1)\backslash\{y^-\} \subseteq (u_k\overrightarrow{C}u_1)$.

Subcase 3 $N(y_1) \cap (u_k\overrightarrow{C}u_1) = \emptyset$.

If not, we may choose $z \in N(y_1) \cap (u_k\overrightarrow{C}u_1)$, such that $N(y_1) \cap (z^+\overrightarrow{C}u_1) = \emptyset$. We define a mapping f_3 on $V(C)$ as follows:

$$f_3(v) = \begin{cases} v^- & \text{if } v \in y^+\overrightarrow{C}v_k \cup u_k^+\overrightarrow{C}z^+; \\ v^+ & \text{if } v \in z^{+2}\overrightarrow{C}y^{-2}; \\ z^+ & \text{if } v = u_k; \\ v_k & \text{if } v = y; \\ z^{+2} & \text{if } v = y^-. \end{cases}$$

Using a similar argument as above , we have $|f_3(N_C(x, u_k))| \geq NC2(G)$. Moreover, $z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k)), N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. In fact, clearly, $z^+ \notin N_C(y_1, z^+)$, if $z^+ \in f_3(N_C(x, u_k))$, then $u_k \in N_C(x, u_k)$, a contradiction; if $u_1^+ \in N_C(y_1, z^+)$, then $u_1^+ \in N(z^+)$ since $N_C(y_1) \cap (y^{-2}\overrightarrow{C}u_k) = \emptyset$, so $C' = x_1y\overrightarrow{C}zy_1y^-\overleftarrow{C}u_1^+z^+\overrightarrow{C}u_1x_1$ is a D –cycle longer than C, a contradiction; if $u_1^+ \in f_3(N_C(x, u_k))$ then $u_1 \in N_C(x, u_k)$,a contradiction; so we have $z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k))$. Suppose $s \in N_C(y_1, z^+) \cap f_3(N_C(x, u_k))\backslash\{y^-, w_k\}$. Now, we consider the following cases.

(i) If $s \in v_k\overrightarrow{C}z^+$, then We have $s^+u_k \in E(G)$ since $N(x) \cap (u_k\overrightarrow{C}w_k) = \emptyset$. Put

$$C' = \begin{cases} xv_k\overleftarrow{C}yx_1u_1\overrightarrow{C}y^-y_1s\overleftarrow{C}u_ks^+\overrightarrow{C}v_1x & \text{if } y_1s \in E(G); \\ xv_k\overleftarrow{C}yx_1u_1\overrightarrow{C}y^-y_1z\overleftarrow{C}s^+u_k\overrightarrow{C}sz^+\overrightarrow{C}v_1x & \text{if } z^+s \in E(G). \end{cases}$$

(ii) If $s \in z^{+2}\overrightarrow{C}w_k^-$, then we have $s^-u_k, sz^+ \in E(G)$ since $N(x) \cap (u_k\overrightarrow{C}w_k) = N(y_1) \cap (z^+\overrightarrow{C}v_1) = \emptyset$. Put

$$C' = xv_k\overrightarrow{C}yx_1u_1\overrightarrow{C}y^-y_1z\overleftarrow{C}u_ks^-\overleftarrow{C}z^+s\overrightarrow{C}v_1x$$

.

(iii) If $s \in u_1\overrightarrow{C}y^{-2}$, then we have $sz^+ \in E(G)$ since $N(y_1) \cap (u_1\overrightarrow{C}y^{-2}) = \emptyset$. Put

$$C' = \begin{cases} xs^{-}\overleftarrow{C}u_1x_1y\overrightarrow{C}zy_1y^{-}\overleftarrow{C}sz^{+}\overrightarrow{C}v_1x & \text{if } xs^{-} \in E(G); \\ xv_k\overleftarrow{C}yx_1u_1\overleftarrow{C}s^{-}u_k\overrightarrow{C}zy_1y^{-}\overleftarrow{C}sz^{+}\overrightarrow{C}v_1x & \text{if } u_ks^{-} \in E(G). \end{cases}$$

(iv) If $s \in y\overrightarrow{C}v_k$, then we have $sz^{+} \in E(G)$ since $N(y_1) \cap (y\overrightarrow{C}v_k) = \emptyset$.Put

$$C' = \begin{cases} xs^{+}\overrightarrow{C}zy_1y^{-}\overleftarrow{C}u_1x_1y\overrightarrow{C}sz^{+}\overrightarrow{C}v_1x & \text{if } xs^{+} \in E(G); \\ xv_k\overleftarrow{C}s^{+}u_k\overrightarrow{C}zy_1y^{-}\overleftarrow{C}u_1x_1y\overrightarrow{C}sz^{+}\overrightarrow{C}v_1x & \text{if } u_ks^{+} \in E(G). \end{cases}$$

In any cases, C' is a D–cycle longer than C, a contradiction. Therefore we have $N_C(y_1, z^{+}) \cap f_3(N_C(x, u_k)) \subseteq \{y^{-}, w_k\}$. So we have

$$\begin{aligned} |V(C)| &\geq |N_C(y_1, z^{+}) \cup f_3(N_C(x, u_k)| + 2 \\ &\geq |N_C(y_1, z^{+}| + |N_C(x, u_k)| - 2 + 2 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$. Hence, $N(y_1) \cap (u_k\overrightarrow{C}v_1) = \emptyset$.

Thus, $N(y_1) = \{y^{-}\}$, which contradicts to $\delta \geq 2$. Therefore, we know that $N_R(y^{-}) = \emptyset$.

So we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^{-})| + 1 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^{-})| - 2 + 1 \\ &= |N(x, u_k) \backslash N_R(x, u_k)| + |N(x_1, y^{-}) \backslash N_R(x_1, y^{-})| - 1 \\ &= |N(x, u_k)| + |N(x_1, y^{-})| - 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So we have $N(x_1) \cap (u_1^{+}\overrightarrow{C}v_k) = \emptyset$, hence, $N(x_1) \subseteq u_k\overrightarrow{C}u_1$.

Case 2 $N(x_1) \cap (u_k\overrightarrow{C}v_1) = \emptyset$.

Otherwise, since $v_1x_1 \notin E(G)$, we can choose $y, y \in u_k\overrightarrow{C}w_k$, such that $N(x_1) \cap (y^{+}\overrightarrow{C}v_1) = \emptyset$. Therefore, we can define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^{-} & \text{if } v \in u_1^{+}\overrightarrow{C}y; \\ v^{+} & \text{if } v \in y^{+}\overrightarrow{C}w_k; \\ y^{+} & \text{if } v = u_1, \\ y & \text{if } v = v_1. \end{cases}$$

Using a similar argument as before, we have $|g(N_C(x, u_k))| \geq NC2(G)$, $y^{+} \notin g(N_C(x, u_k)) \cup N(x_1, y^{+})$ and $g(N_C(x, u_k)) \cap N(x_1, y^{+}) \subseteq \{u_1\}$. Hence, by Claim 6 we have

$$\begin{aligned} |V(C)| &\geq |g(N_C(x, u_k)) \cup N(x_1, y^{+})| + 1 \\ &\geq |g(N_C(x, u_k))| + |N(x_1, y^{+})| - 1 + 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So $N(x_1)\cap(u_k\overrightarrow{C}v_1)=\emptyset$. Then $N(x_1)=\{u_1\}$, which contradicts to $\delta\geq 2$.

Claim 8 If $x_1\in N_R(u_1)$ and $N(x_1)\cap(u_1^+\overrightarrow{C}v_k)\neq\emptyset$, then $|\{u_ku_1^+,y^-w_k\}\cap E(G)|=1$ for $y\in N(x_1)\cap(u_1^+\overrightarrow{C}v_k)$ with $N(x_1)\cap(u_1^+\overrightarrow{C}y^-)=\emptyset$.

First we have $d(x_1,y^-)=2$ and $|N(x_1,y^-)|\geq NC2(G)$.Let $u_ku_1^+\notin E(G)$. Now we define a mapping f on $V(C)$ as follows:

$$f(v)=\begin{cases} v^- & \text{if } v\in u_k^{+2}\overrightarrow{C}v_1\cup u_1^{+2}\overrightarrow{C}y^-;\\ v^+ & \text{if } v\in y\overrightarrow{C}w_{k-1};\\ y^- & \text{if } v=u_k;\\ y & \text{if } v=v_k;\\ u_1 & \text{if } v=u_k^+;\\ v_1 & \text{if } v=u_1^+;\\ u_k & \text{if } v=u_1.\end{cases}$$

Then $|f(N_C(x,u_k))|=|N_C(x,u_k)|\geq NC2(G)-1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $f(N_C(x,u_k))\cap N(x_1,y^-)\subseteq\{w_k,u_1,y\}$. But we have $y^-,v_1,u_k\notin f(N_C(x,u_k))\cup N(x_1,y^-)$ by the choice of y Claims 2 and 4, respectively. Therefore, by Claim 5 we have

$$\begin{aligned}|V(C)| &\geq |f(N_C(x,u_k))\cup N_C(x_1,y^-)|+3\\ &\geq |f(N_C(x,u_k))|+|N_C(x_1,y^-)|-3+3\\ &\geq 2NC2(G)-2.\end{aligned}$$

So $V(C)=f(N_C(x,u_k))\cup N_C(x_1,y^-)\cup\{v_1,y^-,u_k\}$ by the assumption on $|V(C)|$, and in particular, $f(N_C(x,u_k))\cap N_C(x_1,y^-)=\{w_k,u_1,y\}$. Therefore, $y^-w_k\in E(G)$. Using a similar argument as above, we have if $y^-w_k\notin E(G)$, then $u_ku_1^+\in E(G)$.

Claim 9 There exists a vertex x with $x\notin V(C)$ such that $N_R(u_i)=N_R(w_i)=\emptyset$.

We only prove $N_R(u_i)=\emptyset$. If not, we may choose $x\notin V(C)$ such that $\min\{t_i\}$ is as small as possible. By Claim 7, without loss of generality, suppose that $t_k=\min\{t_i\}$ for the vertex x. Let $x_1\in N_R(u_1),x_2\in N_R(u_k)$. By Claims 2 and 3, $x\neq x_1,x_2;x_1\neq x_2$. And by Claim 5 and the choice of x, we have $N(x_i)\cap(u_k\overrightarrow{C}v_1)=\emptyset$, for $i=1,2$. Since $\delta\geq 2$, $N(x_1)\cap(u_1^+\overrightarrow{C}v_k)\neq\emptyset$. Choose $y\in N(x_1)\cap(u_k\overrightarrow{C}v_k)$ such that $N(x_1)\cap(u_1^+\overrightarrow{C}y^-)=\emptyset$, then $d(x_1,y^-)=2$ and $|N(x_1,y^-)|\geq NC2(G)$. By Claim 8, we have $u_ku_1^+$ or $y^-w_k\in E(G)$.

First we prove that $N(x_2)\cap(y\overrightarrow{C}v_k)=\emptyset$. If not, we may choose $z\in y^+\overrightarrow{C}v_k^-$ such that $N(x_2)\cap(z^+\overrightarrow{C}v_k)=\emptyset$ by Claim 5. Then $d(x_2,z^+)=2$ and $|N(x_2,z^+)|\geq NC2(G)$. Now we define a mapping f on $V(C)$ as follows:

$$f(v)=\begin{cases} v^- & \text{if } v\in u_1^+\overrightarrow{C}y^-\cup z^{+2}\overrightarrow{C}v_k;\\ v^+ & \text{if } v\in y\overrightarrow{C}z^-\cup u_k\overrightarrow{C}w_k;\\ y & \text{if } v=z;\\ v_k & \text{if } v=z^+;\\ u_k & \text{if } v=v_1;\\ y^- & \text{if } v=u_1.\end{cases}$$

Then $|f(N_C(x_2,z^+))|=|N_C(x_2,z^+)|\geq NC2(G)-1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $f(N_C(x_2,z^+))\cap N(x_1,y^-)\subseteq\{u_1,y\}$. But $y^-,v_k,v_1\notin f(N_C(x_2,z^+))\cup N(x_1,y^-)$, otherwise, $u_1z^+\in E(G)$ or $y^-v_k\in E(G)$ or $z^+w_k\in E(G)$ by Claim 5, and hence the D-cycle

$$C'=\begin{cases} u_1\overrightarrow{C}zx_2u_k\overrightarrow{C}v_1xv_k\overleftarrow{C}z^+u_1 & \text{if } u_1z^+\in E(G);\\ u_1x_1y\overrightarrow{C}v_ky^-\overleftarrow{C}u_1^+u_k\overrightarrow{C}u_1 & \text{if } y^-v_k\in E(G);\\ xv_k\overleftarrow{C}z^+w_k\overleftarrow{C}u_kx_2z\overleftarrow{C}v_1x & \text{if } z^+w_k\in E(G).\end{cases}$$

is longer than C, a contradiction. Therefore, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x_2,z^+))\cup N_C(x_1,y^-)|+3\\ &\geq |f(N_C(x_2,z^+))|+|N_C(x_1,y^-)|-2+3\\ &\geq 2NC2(G)-1,\end{aligned}$$

which contradicts to that $|V(C)|\leq 2NC2(G)-2$. So we have $N(x_2)\cap(y\overrightarrow{C}v_k)=\emptyset$. Hence $N(x_2)_(u_1^+\overrightarrow{C}y^-)\cup\{u_k\}$.

Now, we prove that $N(x_2)\cap(u_1^+\overrightarrow{C}y^-)=\emptyset$. In fact, we may choose $z\in u_1^+\overrightarrow{C}y^{-2}$ with $z\in N(x_2)$ such that $N(x_2)\cap(u_1^+\overrightarrow{C}z^-)=\emptyset$. (Since $x_2y^-\notin E(G)$, otherwise, $C'=u_1\overrightarrow{C}y^-x_2u_k\overrightarrow{C}v_1xv_k\overleftarrow{C}yx_1u_1$ is a D-cycle longer than C, a contradiction.) Then $d(x_2,z^-)=2$ and $|N(x_2,z^-)|\geq NC2(G)$. We define a mapping g on $V(C)$ as follows:

$$g(v)=\begin{cases} v^- & \text{if } v\in z^+\overrightarrow{C}v_k;\\ v^+ & \text{if } v\in u_k\overrightarrow{C}z^{-2};\\ v_k & \text{if } v=z;\\ u_k & \text{if } v=z^-.\end{cases}$$

Then we have $|g(N_C(x_2,z^-))|\geq NC2(G)-1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $g(N_C(x_2,z^-))\cap N(x_1,y^-)\subseteq\{u_1\}$. But $v_1,u_k\notin g(N_C(x_2,z^-))\cup N(x_1,y^-)$, otherwise since $u_k\notin g(N_C(x_2,z^-))\cup N(x_1,y^-)$, $w_kz^-\in E(G)$ by Claims 2 and 4, and hence the D-cycle $u_1\overrightarrow{C}z^-w_k\overleftarrow{C}\ u_kx_2z\overrightarrow{C}v_kxv_1u_1$ is longer than C, a contradiction. Therefore, by Claim 5 we have

$$\begin{aligned}|V(C)| &\geq |g(N_C(x_2,z^-))\cap N(x_1,y^-)|+2\\ &\geq |g(N_C(x_2,z^-))|+|N(x_1,y^-)|-1+2\\ &\geq 2NC2(G)-1,\end{aligned}$$

which contradicts to that $|V(C)|\leq 2NC2(G)-2$. So we have $N(x_2)\cap(u_1^+\overrightarrow{C}y^-)=\emptyset$.

Therefore, $N(x_2)=\{u_k\}$, which contradicts to $\delta\geq 2$.

Claim 10 For any $x\notin V(C), t_i\geq 3$.

Otherwise, there exists a vertex $x, x\notin V(C)$, such that $\min\{t_i\}=2$ by Claim 6. Note that the choice of the vertex x in Claim 9, we have $N_R(u_i)=N_R(w_i)=\emptyset$ for the vertex x. Without loss of generality, suppose $t_1=2$, then $N_C^-(u_1)\cap N_C(w_1)=\{u_1\}$ by Claim 4, $N(x)\cap N^+(x)=\emptyset$ by Claim 2, and $N_C^-(u_1)\cap N(x)=N^-(x)\cap N_C(w_1)=\emptyset$ by Claim 3. Hence, $N_C^-(x,u_1)\cap N_C(x,w_1)=\{u_1\}$. We also have $|N_C(x,u_1)|\geq NC2(G)$ and $|N_C(x,w_1)|\geq NC2(G)$ since $d(x,u_1)=d(x,w_1)=2$. Then

$$\begin{aligned}|V(C)| &\geq |N_C^-(x,u_1)\cup N_C(x,w_1)|\\ &\geq |N_C^-(x,u_1)|+|N_C(x,w_1)|-1\\ &\geq 2NC2(G)-1,\end{aligned}$$

which contradicts to that $|V(C)|\leq 2NC2(G)-2$.

By Claim 10, we have $|V(C)|=k+\sum\limits_{i=1}^{k}t_i\geq 4k$. Thus we get the following.

Claim 11 For any $x, x\notin V(C)$,

$$d(x)\leq\frac{|V(C)|}{4}\leq\frac{2NC2(G)-2}{4}=(NC2(G)-1)/2.$$

Claim 12 $u_i^+u_j\notin E(G)$, for the vertex x as in Claim 9.

In fact, if $u_i^+u_j\in E(G)$, then the cycle $u_i^+\overrightarrow{C}v_jxv_i\overleftarrow{C}u_ju_i^+$ is a longest D-cycle not containing u_i, by Claim 9. Thus $d(u_i)\leq(NC2(G)-1)/2$ by Claim 11. So we have

$$NC2(G)\leq|N(x,u_i)|\leq d(x)+d(u_i)\leq NC2(G)-1,$$

a contradiction. We choose x as in Claim 9, and define a mapping f on $V(C)$ as follows:

$$f(v)=\begin{cases}v^+ & \text{if } v\in u_1\overrightarrow{C}v_k^-;\\ v^- & \text{if } v\in u_k^+\overrightarrow{C}v_1;\\ u_1 & \text{if } v=v_k;\\ v_1 & \text{if } v=u_k.\end{cases}$$

Then $|f(N_C(x,u_k))|\geq NC2(G)$ and $|N_C(x,u_1)|\geq NC2(G)$ by Claim 10. Moreover, we have $f(N_C(x,u_k))\cap N_C(x,u_1)_{\{}v_2,v_3,\ldots,v_k,w_k\}$. By Claims 2, 4, and 12, we also have $u_2^+,u_3^+,\ldots,u_{k-1}^+\notin f(N_C(x,u_k))\cup N_C(x,u_1)$. Therefore, we have

$$
\begin{aligned}
|V(C)| &\geq |f(N_C(x,u_k)) \cup N_C(x,u_1)| + k - 2 \\
&\geq |f(N_C(x,u_k))| + |N_C(x,u_1)| - k + k - 2 \\
&\geq 2NC2(G) - 2.
\end{aligned}
$$

So

$$V(C) = f(N_C(x,u_k)) \cup N_C(x,u_1) \cup \{u_2^+, u_3^+, \ldots, u_{k-1}^+\}$$

by the assumption on $|V(C)|$, and in particular,

$$f(N_C(x,u_k)) \cap N_C(x,u_1) = \{v_2, v_3, \ldots, v_k, w_k\}.$$

Then $u_1 w_k, u_k w_{k-1} \in E(G)$.

Claim 13 $k = 2$.

If there exists $v \in V(C)\backslash\{v_1, v_k\}$, by partition of $V(C)$, we have $v^{+2} \in f(N_C(x,u_k)) \cup N_C(x,u_1) \cup \{u_2^+, u_3^+, ..., u_{k-1}^+\}$. If $v^{+2} \in N_C(x,u_1)$, then $v^{+2}u_1 \in E(G)$, and the cycle $u_1 v^{+2} \overrightarrow{C} v_1 x v \overleftarrow{C} u_1$ is a D-cycle not containing v^+ by Claim 9. Thus $d(v^+) \leq (NC2(G)-1)/2$ by Claim 11. So we have

$$NC2(G) \leq |N(x,v^+)| \leq d(x) + d(v^+) \leq NC2(G) - 1,$$

a contradiction.So $v^+ \in N(x,u_k)$, which contradicts to Claims 2,3. Hence we have $k = 2$.

Claim 14 Each of the followings does not hold :

(1) There is $u \in u_1 \overrightarrow{C} v_2$, such that $u^+u_1 \in E(G)$ and $u^-u_2 \in E(G)$.
(2) There is $u \in u_2 \overrightarrow{C} v_1$, such that $u^-u_1 \in E(G)$ and $u^+u_2 \in E(G)$.
(3) There is $u \in u_2 \overrightarrow{C} v_1$, such that $u^+w_1 \in E(G)$ and $u^-w_2 \in E(G)$.
(4) There is $u \in u_1 \overrightarrow{C} v_2$, such that $u^+w_2 \in E(G)$ and $u^-w_1 \in E(G)$.

If not, suppose there is $u \in u_1 \overrightarrow{C} v_2$, such that $u^+u_1 \in E(G)$ and $u^-u_2 \in E(G)$. We define a mapping h on $V(C)$ as follows :

$$h(v) = \begin{cases} v^+ & \text{if } v \in u_1 \overrightarrow{C} u^- u_2 \cup u^+ \overrightarrow{C} w_1; \\ v^- & \text{if } v \in u_2^+ \overrightarrow{C} v_1; \\ u^+ & \text{if } v = v_2; \\ v_1 & \text{if } v = u_2; \\ u_1 & \text{if } v = u; \\ u & \text{if } v = u_2^+. \end{cases}$$

Then $|h(N_C(x,u_2))| \geq NC2(G)$ and $|N_C(x,u_1)| \geq NC2(G)$. Moreover we have $u_1 \notin N(x,u_1) \cup h(N(x,u_2))$, and $N(x,u_1) \cap h(N(x,u_2)) \subseteq \{v_2, u^+\}$. In fact, clearly $u_1 \notin N(x,u_1)$, if $u_1 \in h(N(x,u_2))$, then $u \in N(x,u_2)$, a contradiction. Let $s \in N(x,u_1) \cap h(N(x,u_2))\backslash\{v_2,u^+\}$, if $s \in u_1^+ \overrightarrow{C} v_2 \cap N(x,u_1) \cap h(N(x,u_2))\backslash\{v_2,u^+\}$ then $su_1 \in E(G)$ and $s^-u_2 \in E(G)$; or if

$s \in u_2\overrightarrow{C}w_2 \cap N(x,u_1) \cap h(N(x,u_2))$, then $su_1 \in E(G)$ and $s^+u_2 \in E(G)$, both cases contradict to Claim 3. So $u_1 \notin N(x,u_1) \cup h(N(x,u_2))$, $N(x,u_1) \cap h(N(x,u_2)) \subseteq \{v_2, u^+\}$. Hence

$$\begin{aligned} |V(C)| &\geq |h(N_C(x,u_2)) \cup N_C(x,u_1)| + 1 \\ &\geq |h(N_C(x,u_2))| + |N_C(x,u_1)| - 2 + 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. Similarly, (2), (3) and (4) are true.

Claim 15 $N(u_2) \cap (u_1\overrightarrow{C}w_1^-) = N(u_1) \cap (u_2\overrightarrow{C}w_2^-) = \emptyset$.

If not, we may choose $z \in N(u_2) \cap (u_1\overrightarrow{C}w_1^-)$, such that $N(u_2) \cap (u_1\overrightarrow{C}z^-) = \emptyset$. then $u_1z \in E(G)$ (if not, $u_1z \notin E(G)$ then $u_2z^- \in E(G)$ by partition of $V(G)$, which contradicts the choice of z) and $N(u_1) \cap (z^+\overrightarrow{C}w_1) = \emptyset$ (if not, we may choose $s \in N(u_1) \cap (z^+\overrightarrow{C}w_1)$, such that $N(u_1) \cap (z^+\overrightarrow{C}s^-) = \emptyset$ since $z^+u_1 \notin E(G)$. So $s^-u_1 \notin E(G)$,by partition of the $V(C)$, $s^{-2}u_2 \in E(G)$. Which contradicts Claim 14) Moreover $u_1^+\overrightarrow{C}z \subseteq N(u_1)$, and $z\overrightarrow{C}v_2 \subseteq N(u_2)$. Similarly , we have $y \in u_2\overrightarrow{C}w_2$,such that $u_2y, u_1y \in E(G)$ and $N(u_1) \cap (u_2\overrightarrow{C}y^-) = N(u_2) \cap (y^+\overrightarrow{C}w_2) = \emptyset$, $y\overrightarrow{C}v_1 \subseteq N(u_1)$ and $u_2^+\overrightarrow{C}y \subseteq N(u_2)$.

Now we define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^+ & \text{if } v \in v_2\overrightarrow{C}w_2^-; \\ v^- & \text{if } v \in u_1\overrightarrow{C}w_1; \\ v_2 & \text{if } v = w_2; \\ w_1 & \text{if } v = v_1. \end{cases}$$

Using similar argument as above , consider $N(x,w_1) \cup g(N(x,w_2))$, there exists $u \in V(C)$, such that $w_1u, w_2u \in E(G)$. Without loss generality, we may assume $u \in u_1\overrightarrow{C}w_1$, Moreover then $N(w_2) \cap (u^+\overrightarrow{C}w_1) = N(w_1) \cap (u_1\overrightarrow{C}u^-) = \emptyset$, and $v_1\overrightarrow{C}u \subseteq N(w_2)$, $u\overrightarrow{C}v_2 \subseteq N(w_1)$. Let $u \neq z$. If $u \in z\overrightarrow{C}w_1^-$, $u^-u_2 \in E(G)$ by partition of $V(C)$ since $uu_1 \notin E(G)$, which contradicts to Claim 4 ; if $u \in u_1\overrightarrow{C}z$, then $C' = xv_2w_1u\overrightarrow{C}w_1^-u_2\overrightarrow{C}w_2u^-\overleftarrow{C}v_1x$ is a D-cycle longer than C, a contradiction. If $u = z$, since $z^{+2}u_1 \notin E(G), z^+u_2 \in E(G)$ by partition of $V(C)$, which contradicts to Claim 4. Hence $N(u_2) \cap (u_1\overrightarrow{C}w_2^-) = \emptyset$. Similarly $N(u_1) \cap (u_2\overrightarrow{C}w_1^-) = \emptyset$.

By Claim 15 we have

Claim 16 If there exists $z \in v_1\overrightarrow{C}v_2$, such that $u_2z \in E(G)$, then $u_1z \in E(G)$ and $u_1^+\overrightarrow{C}z \subseteq N(u_1)$, $z\overrightarrow{C}w_1 \subseteq N(u_2)$. similarly if there exists $z \in v_2\overrightarrow{C}v_1$, such that $u_2z \in E(G)$, then $u_1z \in E(G)$ and $u_2^+\overrightarrow{C}z \subseteq N(u_2)$, $z\overrightarrow{C}w_2 \subseteq N(u_1)$.

Proof of Theorem 5

Now we are going to complete the proof of Theorem 5. We choose x as in Claim 9. By Claim 13, we know that $k = 2$.

First we prove that there exists $u \in V(C)$ such that $u_1, u_2 \in N(u)$. If there is not any $u \in V(C)\backslash\{v_2, w_1, u_2^+\}$ such that $u_2u \notin E(G)$, then $w_1^-u_1 \in E(G)$ (if not, $w_1^{-2}u_2 \in E(G)$ by

partition of $V(C)$). If $u_1w_1 \notin E(G)$ then $u_2w_1^- \in E(G)$, so we have $u_1, u_2 \in N(w_1^-)$; if there is $u \in V(C)$, such that $u_2u \in E(G)$ then, by Claim 16, $u_1u \in E(G)$, hence $u_1, u_2 \in N(u)$.

By Claim 16, clearly, there are not $z \in u_1\overrightarrow{C}w_1, y \in u_2\overrightarrow{C}w_2$, such that $yz \in E(G)$.

So we have $G \in \mathcal{J}_1$. The proof of Theorem 5 is finished.

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References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York(1976).

[2] H.J. Broersma and H.J. Veldman, *Long dominating cycles and paths in graphs with large neighborhood unions*, Journal of Graph Theory 15(1991), 29-38.

[3] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.

[4] R.Q.Shen and F.Tian, *Long dominating cycles in graphs*, *Discrete Mathematics*, Vol.177(1997), 287-294.

[5] F. Tian and L.Z.Zhang *Long dominating cycle in a kind of 2-connected graphs*, Systems Science and Mathematical Science 8(1995), 66-74.

[6] Z.R.Sun, *Long dominating cycle in graphs* (Submitted).

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The Crossing Number of the Join of C_m and P_n

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Abstract: In this paper, the crossing numbers of $P_m \vee P_n$, $C_m \vee P_n$ and $C_m \vee C_n$ are determined for arbitrary integers $m, n \geq 1$, which are related with parallel bundles in planar map geometries, i.e., Smarandache spherical geometries.

Keywords: Crossing number, join of graphs, path, cycle.

AMS(2000): O5C25, O5C62.

§1. Introduction

Let G be a simple and undirected graph with vertex set V and edge set E. The *crossing number* $cr(G)$ of the graph G is the minimum number of pairwise intersections of edges in all drawings of G in a plane, which are related with parallel bundles in *planar map geometries*, i.e., *Smarandache spherical geometries* (see [6]-[7] for details). It is well known that the crossing number of a graph is attained only in *good* drawings , means that no edge crosses itself, no two edges cross more than once, no two edges incident with the same vertex cross, no more than two edges cross at a point of the plane, and no edge meets a vertex which is not one of its endpoints. It is easy to see that a drawing with the minimum number of crossings (an *optimal drawing*) is always a good drawing. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$. Let A and B be disjoint edge subsets of G. We denote by $cr_D(A, B)$ the number of crossings between edges of A and B, and by $cr_D(A)$ the number of crossings whose two crossed edges are both in A. Let H be a subgraph of G, the restricted drawing $D|_H$ is said to be a *subdrawing* of H. As for more on the theory of crossing number, we refer readers to [1] and [2]. In this paper, we also use the term *region* in non-planar drawings. In this case, crossings are considered to be vertices of the map.

Let G_1 and G_2 be two disjoint graphs. The *union* of G_1 and G_2, denoted by $G_1 + G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and the *join* of G_1 and G_2 is obtained by adjoining every vertex of G_1 to every vertex of G_2 in $G_1 + G_2$ which is denoted by $G_1 \vee G_2$ (see [3]).

Let $K_{m,n}$ denote the complete bipartite graph on sets of m and n vertices, P_n the path of length n and C_m the cycle with m vertices.

From these definitions, following results are well-known.

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Proposition 1.1 *Let G_1 be a graph homeomorphic to G_2. Then $cr(G_1) = cr(G_2)$.*

Proposition 1.2 *If G_1 is a subgraph of G_2, then $cr(G_1) \leq cr(G_2)$.*

Proposition 1.3 *Let D be a good drawing of a graph G. If A, B and C are three mutually disjoint edge subsets of G, then we have*

(1) $cr_D(A \cup B) = cr_D(A) + cr_D(A, B) + cr_D(B)$;

(2) $cr_D(A \cup B, C) = cr_D(A, C) + cr_D(B, C)$.

Proposition 1.4([4]) *If G has n vertices and m edges with $n \geq 3$, then $cr(G) \geq m - 3n + 6$.*

Computing the crossing number of graphs is a classical problem , and yet it is also an elusive one. In fact, Garey and Johnson in [5] have proved that to determine the crossing number of graphs is NP-complete in general. At present, the classes of graphs whose crossing numbers have been determined are very scarce.

On the crossing number of the complete bipartite graphs $K_{m,n}$, Zarankiewicz gave a drawing of $K_{m,n}$ in [8] which demonstrates that

$$cr(K_{m,n}) \leq Z(m, n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor.$$

and conjectured $cr(K_{m,n}) = Z(m, n)$, Which is called the *Zarankiewicz conjecture*. More precisely, Kleitman proved in [9] that if $m \leq 6$ and $m \leq n$, $cr(K_{m,n}) = Z(m, n)$.

As we known, results for the join of graphs are fewer, particularly, Bogdan Oporowski proved $cr(C_3 \vee C_5) = 6$ in [4]. Based on this, we begin to consider the crossing numbers of the join of P_m and P_n, C_m and P_n, C_m and C_n, and get the following theorems which consist of these main results in this paper.

Theorem A *If $m \geq 1, n \geq 1$ and $min\{m, n\} \leq 5$, then*

$$cr(P_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor.$$

If $m \geq 3, n \geq 1$ and $min\{m, n+1\} \leq 6$, then

$$cr(C_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor + 1$$

and if $m \geq 3, n \geq 3, \quad min\{m, n\} \leq 6$, then

$$cr(C_m \vee C_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2.$$

Theorem B *If the Zarankiewicz conjecture is held for $m \geq 7$ and $m \leq n$, then if $m \geq 1, n \geq 1, \quad min\{m, n\} \geq 6$,*

$$cr(P_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor;$$

if $m \geq 3, n \geq 1, \quad min\{m, n+1\} \geq 7$,

$$cr(C_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor + 1$$

and if $m \geq 3, n \geq 3, \quad min\{m,n\} \geq 7$,

$$cr(C_m \vee C_n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2.$$

§2. Some Lemmas

Lemma 2.1 (1) *There exists a good drawing* D_1 *of* $P_m \vee P_n$ *for given integers* $m \geq 1$ *and* $n \geq 1$ *such that*

$$cr_{D_1}(P_m \vee P_n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor;$$

(2) *There exists a good drawing* D_2 *of* $C_m \vee P_n$ *for given integers* $m \geq 3$ *and* $n \geq 1$ *such that*

$$cr_{D_2}(C_m \vee P_n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor + 1;$$

(3) *There exists a good drawing* D_3 *of* $C_m \vee C_n$ *for given integers* $m \geq 3$ *and* $n \geq 3$ *such that*

$$cr_{D_3}(C_m \vee C_n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2.$$

Proof By Fig.2.1-Fig.2.3, the conclusions are immediately held. □

Lemma 2.2 $\quad cr(C_3 \vee C_3) = 3$.

Proof From Lemma 2.1(3), $cr(C_3 \vee C_3) \leq 3$. We know $C_3 \vee C_3$ has 6 vertices and 15 edges, then $cr(C_3 \vee C_3) \geq 15 - 3 \times 6 + 6 = 3$. Therefore the conclusion is held. □

In the following Lemmas, let G be a connected graph with $V(G) = \{x_1, x_2, \ldots, x_n \ (n \geq 3)\}$ and C_m a cycle with $V(C_m) = \{y_1, y_2, \ldots, y_m\}$. Then we know that $V(C_m \vee G) = V(C_m) \cup V(G)$ and $E(C_m \vee G) = E(C_m) \cup E(G) \cup E^*$, here $E^* = \{x_i y_j | i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\}$.

Lemma 2.3 *For any good drawing* D *of* $C_m \vee G$,

$$cr(C_m \vee G) \geq cr_D(E^*) \geq cr(K_{m,n}).$$

Proof Since the edge-induced subgraph of E^* is $K_{m,n}$, the conclusion is evident. □

Lemma 2.4 *Let* ϕ *be an optimal drawing of* $C_m \vee G$. *Then* $cr_\phi(E(C_m)) = 0$.

Proof We assume there exists an optimal drawing ϕ of $C_m \vee G$ such that $cr_\phi(E(C_m)) \neq 0$. Then $m \geq 4$ and there exist two crossed edges $e, f \in E(C_m)$. We assume that $e = y_i y_j$, $f = y_k y_l$, where i, j, k, l are distinct. For convenience, we denote the crossing between e and f by v. Since C_m is 2-connected, there exist two paths P_1 and P_2 connected y_i and y_k, y_j and y_l, respectively and $P_i (i = 1, 2)$ does not pass v. In the following, we shall produce a new good drawing ϕ' of $C_m \vee G$ (see Fig.2.2 below).

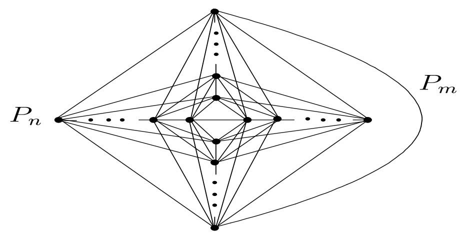

Fig.2.1

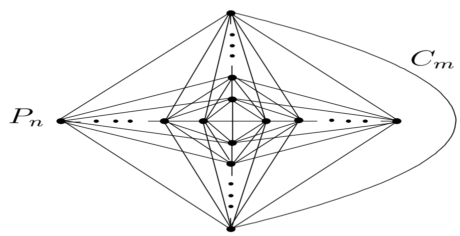

Fig.2.2

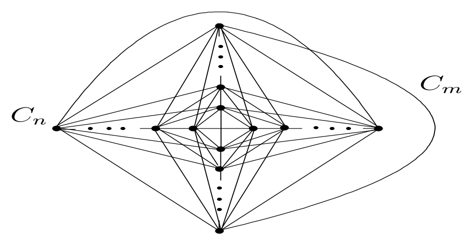

Fig.2.3

At first, we connect y_i to y_l sufficiently close to the section between y_i and v of e and the section between y_l and v of f, then we get a new edge $e' = y_i y_l$. Analogously, we can get another new edge $f' = y_j y_k$. Secondly, we delete two original edges e and f. In this way, we produce a new good drawing ϕ' of $C_m \vee G$ such that the crossing v in ϕ is deleted in ϕ', the other crossings in ϕ are not changed in ϕ' and there is no new crossing occurring in ϕ', then we get that $cr_{\phi'}(C_m \vee G) = cr_\phi(C_m \vee G) - 1$, contradicts to that ϕ is an optimal drawing. $\square$

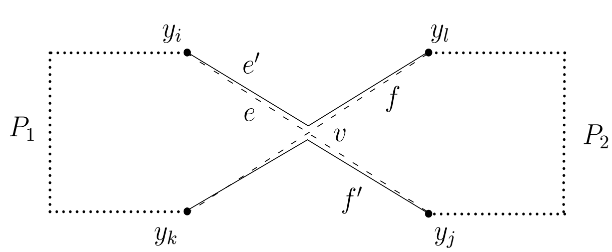

Fig.2.2

Lemma 2.5 *Let ϕ be a good drawing of $C_m \vee G$ such that $cr_\phi(E(C_m)) = 0$, $cr_\phi(E(C_m), E(G)) =$*

0 *and* $cr_\phi(E(C_m), E^*) \leq 1$.

(1) *If* $cr_\phi(E(C_m), E^*) = 0$, *then* $cr_\phi(C_m \vee G) \geq \frac{1}{2}n(n-1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$;

(2) *If* $cr_\phi(E(C_m), E^*) = 1$, *then* $cr_\phi(C_m \vee G) \geq \frac{1}{2}(n-1)(n-2)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$.

Proof Since $cr_\phi(E(C_m)) = 0$, the subdrawing $\phi|_{C_m}$ divides the plane into two regions. As $cr_\phi(E(C_m), E(G)) = 0$ and G is connected, any vertex x_i of G lies in the same region, say the finite region. For convenience, let $E^i = \{x_i y_j | j = 1, 2, \ldots, m\}$ for $i = 1, 2, \ldots, n$. Then $cr_\phi(E^i) = 0$. Since $E^* = \cup_{i=1}^n E^i$, we find that $cr_\phi(E^*) = \sum_{1 \leq i < k \leq n} cr_\phi(E^i, E^k)$.

(i) Since $cr_\phi(E(C_m), E^*) = 0$, then for any $i = 1, 2, \ldots, n$, $x_i y_j$ does not cross any edge in $E(C_m)$. For any integers i, k, $1 \leq i < k \leq n$, x_i must be connected to each y_j $(j = 1, 2, \ldots, m)$, these m edges connecting x_i to all $y_j \in V(C_m)$ which divide the finite region into m subregions, we know that x_k lies in one of these subregions. Thus the m edges connecting x_k to y_j must cross the edges adjacent to x_i at least $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ times (see Fig.2.3 below). Then $cr_\phi(E^i, E^k) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. So $cr_\phi(C_m \vee G) \geq cr_\phi(E^*) = \sum_{1 \leq i < k \leq n} cr_\phi(E^i, E^k) \geq \frac{1}{2}n(n-1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. Our conclusion (1) is held.

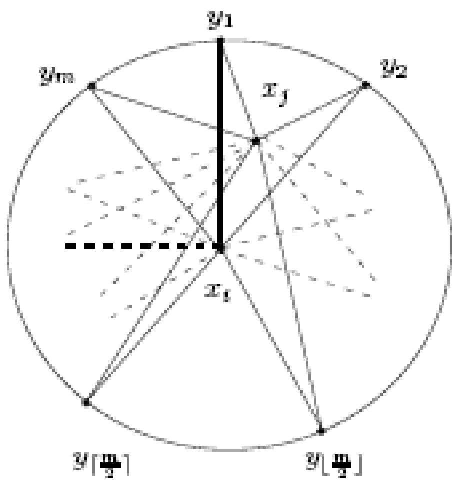

Fig.2.3

(ii) Since $cr_\phi(E(C_m), E^*) = 1$, there exists only one $k \in \{1, 2, \ldots, n\}$. Without loss of generality, we assume that $k = n$ such that for some $j \in \{1, 2, \ldots, m\}$, $x_n y_j$ crosses exactly one edge in $E(C_m)$. For any integer $i = 1, 2, \ldots, n-1$, $x_i y_j$ does not cross any edge in $E(C_m)$. Similar to (i), for $1 \leq i < k \leq n-1$, $cr_\phi(E^i, E^k) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. Then $cr_\phi(C_m \vee G) \geq cr_\phi(E^*) + 1 \geq \sum_{1 \leq i < k \leq n-1} cr_\phi(E^i, E^k) + 1 \geq \frac{1}{2}(n-1)(n-2)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$. Our conclusion (2) is held too. $\square$

§3. Proofs

Proof of Theorem A

(1) If $n = 1$, $P_m \vee P_1$ is a planar graph, the conclusion is held.

If $n \geq 2$, from Lemma 2.1(1) we know $cr(P_m \vee P_n) \leq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$. Since P_n is connected, combining with Lemma 2.3, $cr(P_m \vee P_n) \geq cr(K_{m+1,n+1})$. For $min\{m, n\} \leq 5$, $cr(K_{m+1,n+1}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$. Then $cr(P_m \vee P_n) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$. So the conclusion is held.

(2) From Lemma 2.1(2), we know that $cr(C_m \vee P_n) \leq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor + 1$.

If $n = 1$, $cr(C_m \vee P_1) \leq 1$, and $C_m \vee P_1$ has a subgraph which is homeomorphic to K_5, then the conclusion is held.

If $n \geq 2$, since P_n is connected, combining with Lemma 2.3 and $min\{m, n+1\} \leq 6$, $cr(C_m \vee P_n) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$. We assume there exists an optimal drawing ϕ such that $cr_\phi(C_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$. By Lemma 2.3 and $min\{m, n+1\} \leq 6$, $cr_\phi(E^*) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$. While

$$\begin{aligned} cr_\phi(C_m \vee P_n) &= cr_\phi(E(C_m)) + cr_\phi(E(P_n)) + cr_\phi(E^*) \\ &+ cr_\phi(E(C_m), E(P_n)) + cr_\phi(E(C_m), E^*) + cr_\phi(E(P_n), E^*), \end{aligned}$$

we get $cr_\phi(E(C_m)) = 0$, $cr_\phi(E(C_m), E(P_n)) = 0$ and $cr_\phi(E(C_m), E^*) = 0$, combining with Lemma 2.5(1), $cr_\phi(C_m \vee P_n) \geq \frac{1}{2}n(n+1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. It is easy to check that $\frac{1}{2}n(n+1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor > \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$ for integers $m \geq 3$ and $n \geq 2$, a contradiction. Thus the conclusion is held.

(3) By Lemma 2.2, we have determined the crossing number of $C_3 \vee C_3$. Without loss of generality, we can assume $n \geq 4$ in the following arguments.

From Lemma 2.1(3) we know that $cr(C_m \vee C_n) \leq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2$. Since C_n is connected, by Lemma 2.3 and $min\{m, n\} \leq 6$, $cr(C_m \vee C_n) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. We assume there exists an optimal drawing φ such that

$$\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \leq cr_\varphi(C_m \vee C_n) \leq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 1.$$

By Lemma 2.3 and $min\{m, n\} \leq 6$, $cr_\varphi(E^*) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.
By Lemma 2.4, $cr_\varphi(E(C_m)) = 0$ and $cr_\varphi(E(C_n)) = 0$. As $cr_\varphi(C_m \vee C_n) = cr_\varphi(E(C_m)) + cr_\varphi(E(C_n)) + cr_\varphi(E^*) + cr_\varphi(E(C_m), E(C_n)) + cr_\varphi(E(C_m), E^*) + cr_\varphi(E(C_n), E^*)$, we get that

$$cr_\varphi(E(C_m), E(C_n)) \leq 1, \quad cr_\varphi(E(C_m), E^*) \leq 1.$$

If $cr_\varphi(E(C_m), E(C_n)) = 1$, since C_m and C_n are vertex-disjoint cycles, then they cross at least twice, also a contradiction. So $cr_\varphi(E(C_m), E(C_n)) = 0$.

If $cr_\varphi(E(C_m), E^*) = 0$, by Lemma 2.5(1), $cr_\phi(C_m \vee C_n) \geq \frac{1}{2}n(n-1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. It is easy to check that $\frac{1}{2}n(n-1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor > \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 1$ for integers $m \geq 3$ and $n \geq 4$, a contradiction.

If $cr_\varphi(E(C_m), E^*) = 1$, by Lemma 2.5(2), $cr_\phi(C_m \vee C_n) \geq \frac{1}{2}(n-1)(n-2)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$, it is also easy to check that $\frac{1}{2}(n-1)(n-2)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1 > \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 1$ for $m \geq 3$ and $n \geq 4$, a contradiction too. So the conclusion is held.

This completes the proof of Theorem A. □

Proof of Theorem B

If the Zarankiewicz conjecture is held for integers $m \geq 7$ and $m \leq n$, then the crossing number of $K_{m,n}$ is $Z(m,n)$ for $m \geq 7$ and $m \leq n$, so the proof of Theorem B is analogous to the proof of Theorem A. $\square$

Notice that these drawings D_1, D_2 and D_3 in Fig.2.1 − 2.3 are optimal drawings of $P_m \vee P_n$ for integers $m \geq 1$ and $n \geq 1$, $C_m \vee P_n$ for integers $m \geq 3$ and $C_m \vee C_n$ for integers $m \geq 3$ and $n \geq 3$, respectively.

References

[1] Jonathan L. Gross, Thomas W. Tucker, *Topological Graph Theory*, A Wiley-Interscience Publication, John Wiley & Sons, Canada, 1987.

[2] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, 1969.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, The Macmillan Press LTD., London, 1976.

[4] Bogdan Oporowski and David Zhao, Coloring graphs with crossings, *to appear*.

[5] M.R. Garey and D. S. Johnson, Crossing number is NP-complete, *SIAM J. Algebraic. Discrete Methods*, **4**, (1983), 312-316.

[6] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[7] L.F.Mao, Parallel bundles in planar map geometries, *Scientia Magna*, Vol.1(2005), No.2,120-133.

[8] K. Zarankiewicz, On a problem of P. Turán concerning graphs, *Fund. Math.*, **41**, (1954), 137-145.

[9] D. J. Kleitman, The crossing number of $K_{5,n}$, *J. Combinatorial Theory*, **9**, (1970), 315-323.

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A Note on 4-Ordered Hamiltonicity of Cayley Graphs

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Abstract: A hamiltonian graph G of order n is k-ordered for an integer $k, 2 \leq k \leq n$ if for every sequence $(v_1, v_2, ..., v_k)$ of k distinct vertices of G, there exists a hamiltonian cycle that encounters $(v_1, v_2, ..., v_k)$ in order. For any integer $k \geq 1$, let $G = \mathbb{Z}_{3k-1}$ denote the additive group of integers modulo $3k-1$ and C the subset of $\mathbb{Z}_{3k-1}$ consisting of these elements congruent to 1 modulo 3. Denote by $And(k)$ the Cayley graph $Cay(G : C)$. In this note, we show that $And(k)$ is a 4-ordered hamiltonian graph.

Keywords: Cayley graph, k-ordered, hamiltonicity.

AMS(2000): 05C25

§1. Introduction

All groups and graphs considered in this paper are finite. For any integers $n \geq 3$ and $k, 2 \leq k \leq n$, a hamiltonian graph G of order n is k-ordered if for every sequence $(v_1, v_2, ..., v_k)$ of k distinct vertices of G, there exists a hamiltonian cycle that encounters $(v_1, v_2, ..., v_k)$ in order. Let $G = \mathbb{Z}_{3k-1}$ denote the additive group of integers modulo $3k-1$ with $k \geq 1$ and C the subset of $\mathbb{Z}_{3k-1}$ consisting of these elements congruent to 1 modulo 3. We denote the Cayley graph $Cay(G : C)$ by $And(k)$ in this note.

For $\forall v_i, v_j \in V(And(k))$, $d(v_i)$=$d(v_j)$=k, $v_i \sim v_j$ if and only if $j - i \equiv \pm 1 (\mathrm{mod}\ 3)$. We have known that the diameter of $And(k)$ is 2 and the subgraph of $And(k)$ induced by $\{0, 1, 2, ..., 3(k-1)-2\}$ is $And(k-1)$ by results in references $[2]-[3]$. Therefore, we can get $And(k-1)$ from $And(k)$ by deleting the path $3k-4 \sim 3k-3 \sim 3k-2$. As it has been shown also in [2], there exist 4-regular, 4-ordered graphs of order n for any integer $n \geq 5$. In this note, we research 4-ordered property of $And(k)$.

§2. Main result and its proof

Theorem *$And(k)$ is a 4-ordered hamiltonian graph.*

Proof We have known that $And(k)$ is a hamiltonian graph. For any $S = (x, u, v, w) \subseteq V[And(k)] = \{0, 1, 2, ..., 3k-2\}$, it is obvious that there is a hamiltonian cycle C that encounters the vertices of S, not loss of generality, we can assume it passing through these vertices in the order (x, u, v, w). By a reverse traversing, we also get a hamiltonian cycle that encounters the

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vertices of S in the order (x, w, v, u). Notice that there are six cyclic orders for (x, u, v, w) as follows:

$$(x, u, v, w), \qquad (x, w, v, u);$$
$$(x, w, u, v), \qquad (x, v, u, w);$$
$$(x, u, w, v), \qquad (x, v, w, u).$$

Here, in each row, one is a reversion of another.

Our proof is divided into following discussions.

Firstly, we show that there is a hamiltonian cycle C that encounters the vertices of S in the order (x, v, u, w).

Case 1 $\quad v - u \equiv 0(\text{mod}3)$

Notice that $v-(u-1) = v-u+1 \equiv 1(\text{mod}3)$, $v \sim (u-1)$, $(v+1)-u = v-u+1 \equiv 1(\text{mod}3)$, $(v+1) \sim u$ in this case. There exists a hamiltonian cycle

$$x = 0, 1, 2, 3, ..., u-1, v, v-1, v-2, ..., u, v+1, v+2, ...w, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, v, u, w).

Case 2 $\quad v - u \equiv 1(\text{mod}3)$

In this case, $v-(u-3) = v-u+3 \equiv 1(\text{mod}3)$, $v \sim (u-3)$, $(v+1)-(u-2) = v-u+3 \equiv 1(\text{mod}3)$, $(v+1) \sim (u-2)$. We find a hamiltonian cycle

$$x = 0, 1, 2, ..., u-3, v, v-1, ..., u, u-1, u-2, v+1, v+2, ...w, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, v, u, w).

Case 3 $\quad v - u \equiv 2(\text{mod}3)$

Since $v-(u-2) = v-u+2 \equiv 1(\text{mod}3)$, $v \sim (u-2)$, $(v+1)-(u-1) = v-u+2 \equiv 1(\text{mod}3)$, $(v+1) \sim (u-1)$ in this case. We have a hamiltonian cycle

$$x = 0, 1, 2, ..., u-2, v, v-1, ..., u, u-1, v+1, v+2, ...w, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, v, u, w).

By traversing this cycle in a reverse direction, there is also a hamiltonian cycle that encounters the vertices of S in the order (x, w, u, v).

Next, we show that there is also a hamiltonian cycle C that encounters the vertices of S in the order (x, u, w, v).

Case 1 $\quad w - v \equiv 0(\text{mod}3)$

Notice that $w-(v-1) = w-v+1 \equiv 1(\text{mod}3)$, $w \sim (v-1)$, $(w+1)-v = w-v+1 \equiv 1(\text{mod}3)$, $(w+1) \sim v$ in this case. We find a hamiltonian cycle

$$x = 0, 1, 2, ..., u, ..., v-1, w, w-1, ..., v, w+1, w+2, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, u, w, v).

Case 2 $w - v \equiv 1 \bmod 3$

In this case, $(w+1)-(v-2) = w-v+3 \equiv 1(\bmod 3)$, $(w+1) \sim (v-2)$, $(w+2)-(v-1) = w-v+3 \equiv 1(\bmod 3)$, $(w+2) \sim (v-1)$. There exists a hamiltonian cycle

$$x = 0, 1, 2, ..., u, ..., v-2, w+1, w, w-1, ..., v, v-1, w+2, ..., 3k-2$$

in the graph $And(k)$ encountering vertices of S in the order (x, u, w, v) if $w \neq 3k-2, u \neq v-1$. While $w = 3k-2, u = v-1$, notice that $(3k-2)-v \equiv 1(\bmod 3)$, $3k-v \equiv 0(\bmod 3)$ and $v \equiv 0(\bmod 3)$. So $u+5 = (v-1)+5 = v+4 \equiv 1(\bmod 3)$, $u+5 \sim 0$. The cycle

$$x = 0, 1, 2, 3, ..., u, u+4, u+3, u+2, u+6, u+7, ..., 3k-2, v(u+1), u+5, 0$$

in $And(k)$ is a hamiltonian cycle encountering vertices of S in the order (x, u, w, v).

Case 3 $w - v \equiv 2(\bmod 3)$

By assumption, $(w+1)-(v-1) = w-v+2 \equiv 1(\bmod 3)$, $(w+1) \sim (v-1)$, $(w+2)-v = w-v+2 \equiv 1(\bmod 3)$, $(w+2) \sim v$. We get a hamiltonian cycle

$$x = 0, 1, 2, ..., u, ..., v-1, w+1, w, w-1, ..., v, w+2, ..., 3k-2$$

in $And(k)$ encountering all vertices of S in the order (x, u, w, v) if $w \neq 3k-2, u \neq v-1$. Now if $w = 3k-2, u = v-1$, $w-v \equiv 2(\bmod 3)$, notice that $(w-2)-u = (w-2)-(v-1) = w-v-1 \equiv 1(\bmod 3)$, $(w-2) \sim u$, $w-(u-1) = w-(v-1-1) = w-v+2 \equiv 1(\bmod 3)$, $w \sim (u-1)$, $(w-3)-0 = (3k-2)-3 = 3k-5 \equiv 1(\bmod 3)$, $(w-3) \sim 0$, $(3k-2)-(u+1) = 3k-3-u \equiv 2(\bmod 3)$, $u-0 \equiv 1 \bmod 3)$, $u \sim 0$. There is also a hamiltonian cycle

$$x = 0, u, w-2, w-1, w, u-1, ..., 1, v, v+1, v+2, ..., w-3, 0$$

in $And(k)$ encountering vertices of S in the order (x, u, w, v).

By traversing the cycle in a reverse direction, we also find a hamiltonian cycle that encounters the vertices of S in the order (x, v, w, u).

This completes the proof. □

References

[1] Chris Godsil and Gordon Royle, *Algebraic Graph Theory*Springer2004.

[2] Lenhard Ng, Michelle Schultz, k-ordered hamiltonian graphs, *J.Graph Theory*, Vol.24, No.1, 45-57, 1997.

[3] Faudree R. J., On k-ordered graphs, *J.Graph Theory*, Vol.35, 73-87, 2001.

[4] Bondy J. A. and Murty U. S. R., *Graph Theory with Applications*, The Macmillan Press LTD, 1976.

[5] Kun Cheng and Yongga A, An infinite construction of 3-regular 4-ordered graphs, *J.Inner Mongolia Normal University* (Natural Science), Vol.34, No.2, 158-162, 2005.

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The Crossing Number of Two Cartesian Products

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Abstract: There are several known exact results on the crossing number of Cartesian products of paths, cycles, and complete graphs. In this paper, we find the crossing numbers of Cartesian products of P_n with two special 6-vertex graphs.

Keywords: Cartesian product; Crossing number.

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§1. Introduction

A *drawing* D of a graph G on a surface S consists of an immersion of G in S such that no edge has a vertex as an interior point and no point is an interior point of three edges. We say a drawing of G is a *good drawing* if the following conditions hold:

(1) no edge has a self-intersection;
(2) no two adjacent edges intersect;
(3) no two edges intersect each other more than once;
(4) each intersection of edges is a crossing rather than tangential.

The *crossing number* $cr(G)$ of a graph G is the smallest number of pairs of nonadjacent edges that intersect in a drawing of G in the plane. An *optimal drawing* of a graph G is a drawing whose number of crossings equals $cr(G)$.

Now let G_1 and G_2 be two vertex-disjoint graphs. Then the *union* of G_1 and G_2, denoted by $G_1 \bigcup G_2$, is a graph with $V(G_1 \bigcup G_2) = V(G_1) \bigcup V(G_2)$ and $E(G_1 \bigcup G_2) = E(G_1) \bigcup E(G_2)$. The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{\{(u_i, v_j), (u_h, v_k)\} | (u_i = u_h \text{ and } v_j v_k \in E(G_2)) \text{ or } (v_j = v_k \text{ and } u_i u_h \in E(G_1))\}$. A circuit C of a graph G is called *non-separating* if $G/V(C)$ is connected, and *induced* if the vertex-induced subgraph $G[V(C)]$ of G is C itself. A circuit is called to be an *induced non-separating circuit* if it is both induced and non-separating. For definitions not explained in this paper, readers are referred to [1]. The following result is obvious by definitions.

Lemma 1.1 *If C is an induced non-separating circuit of G, then C must be the boundary of a face in the planar embedding.*

The problem of determining the crossing number of a graph is NP-complete. As we known, the crossing number are known only for a few families of graphs, most of them are Cartesian products of special graphs. For examples,

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$cr(C_3 \times C_3) = 3$ (Harary et al, 1973, see [5]);

$cr(C_3 \times C_n) = n$ (Ringeisen and Beinekein, 1978, see [9]);

$cr(C_4 \times C_4) = 8$ (Dean and Richter, 1995, see [3]);

$cr(C_4 \times C_n) = 2n, \quad cr(K_4 \times C_n) = 3n$ (Beineke and Ringeisen, 1980, see [2])

Let S_{n-1} and P_n be the star and path with n vertices, respectively. Klesc [6] proved that $cr(S_4 \times P_n) = 2(n-2)$ and $cr(S_4 \times C_n) = 2(n-1)$. He also showed that $cr(K_{2,3} \times S_n) = 2n$ [7] and $cr(K_5 \times P_n) = 6n$ in [7]. Peng and Yiew [4] proved that $cr(P_{3,1} \times P_n) = 4(n-1)$.

In this paper, we extend these results to the product $G_j \times P_n, 1 \le j \le 2$ for two special graphs shown in Fig.1 following.

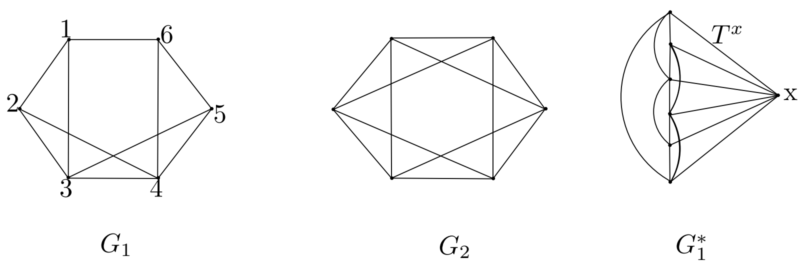


Fig.1

For convenience, we label these six vertices on their outer circuits of G_1 consecutively by integers $1, 2, 3, 4, 5$ and 6 in clockwise, such as those shown in Fig.1. Notice that for any graph $G_i, i = 1, 2$, $G_i \times P_n$ contains n copies of G_i, denoted by $G_i^j (1 \le j \le n)$ and 6 copies of P_n. We call the edges in G_i^j black and the edges in these copies of P_n red. For $j = 1, 2, \cdots n-1$, let $L(j, j+1)$ denote the subgraph of $G_i \times P_n$, induced by six red edges joining G_i^j to G_i^{j+1}. Note that $L(j, j+1)$ is homeomorphic to $6K_2$.

§2. The crossing number of $G_1 \times P_n$

By joining all 6 vertices of G_1 to a new vertex x, we obtain a new graph, denoted by G_1^*. Let T^x be the six edges incident with x, see Fig.1. We know $G_1^* = G_1 \bigcup T^x$ by definition.

Lemma 2.1 $\quad cr(G_1^*) = 2$.

Proof A good drawing of G_1^* shown in Fig.2 following enables us to get $cr(G_1^*) \le 2$. We prove the reverse inequality by a case-by-case analysis. In any good drawing D of G_1^* , there are only three cases, i.e., $cr_D(G_1) = 0$, $cr_D(G_1) = 1$ or $cr_D(G_1) \ge 2$.

Case 1 $\quad cr_D(G_1) = 0$.

Use Euler's formula, $f = 6$ and we note that there are 6 induced non-separating circuits 1231, 2342, 3453, 4564, 12461, 13561. So there are at most 4 vertices of G_1 on each boundary.

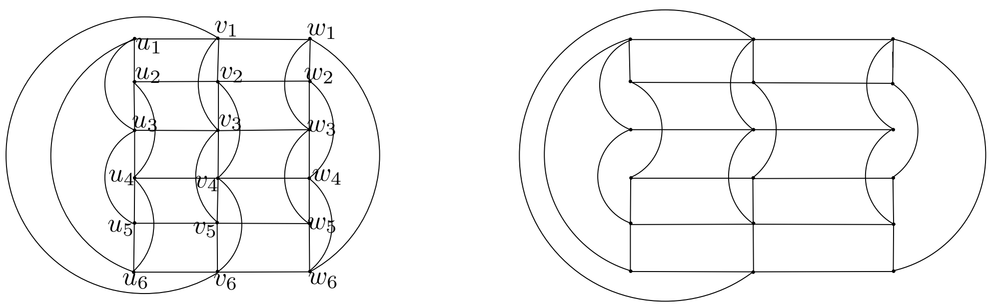
Joining all 6 vertices to x, there are 2 crossings among the edges of G_1 and the edges of T^x at least. This implies $cr(G_1^*) \geq 2$.

Case 2 $\quad cr_D(G_1) = 1$.

There are at most five vertices of G_1 on each boundary. Joining all 6 vertices to x, there are at least one crossing made by edges of G_1 with edges of T^x. So $cr(G_1^*) \geq 2$.

Case 3 $\quad cr_D(G_1) \geq 2$.

Then $cr(G_1^*) \geq 2$. Whence, $cr(G_1^*) = 2$. $\square$



$G_1 \times P_3$ $\qquad\qquad$ $K_{2,3} \times S_2$

Fig.2

Lemma 2.2 *In any good drawing of $G_1 \times P_n$, $n \geq 2$, there are at least two crossings on the edges of G_1^i for $i = 1, 2, \cdots n$.*

Proof Let w_i denote the number of crossings on the edges of G_1^i for $i = 1, 2, \cdots n$ and $H_i = \langle V(G_1^i) \bigcup V(G_1^{i+1}) \rangle_{G_1 \times P_n}$ for $i = 1, 2, \cdots n-1$. First, we prove that $w_n \geq 2$. Let T' be a graph obtained by contracting the edges of G_1^{n-1} in H_{n-1} resulting in a graph homeomorphic to G_1^*.

By the proof of Lemma 2.1, $w_n \geq cr(T') = cr(G_1^*) = 2$. For $i = 1, 2, \cdots n-1$, let T_i be the graph obtained by contracting the edges of G_1^{i+1} in H_i resulting in a graph homeomorphic to G_1^*. Similarly, by Lemma 2.1, we get that $w_i \geq cr(T_i) = cr(G_1^*) = 2$ for $i = 1, 2, \cdots n-1$. $\square$

Lemma 2.3 *If D is a good drawing of $G_1 \times P_n$ in which every copy of G_1 has at most three crossings on its edges, then D has at least $4(n-1)$ crossings.*

Proof Let D be a good drawing of $G_1 \times P_n$ in which every copy of G_1 has at most three crossings on its edges. We first show that in D no black edges of G_1^i cross any black edges of G_1^j for $i \neq j$. If not, suppose there is a black edge of G_1^i crossing with a black edge of G_1^j. Since D is a good drawing and every edge of G_1 is an edge of a cycle, there exists a cycle induced by $V(G_1^i)$ which contains a black edge crossing with at least two black edges of G_1^j. Now delete the black edges of G_1^i. The resulting graph is either

(1) homeomorphic to $G_1 \times P_{n-1}$ for $i = 2, 3, \cdots n-1$; or

(2) contains a subgraph homeomorphic to $G_1 \times P_{n-1}$ for $i = 1$ or $i = n$.

Since every copy of G_1 in $G_1 \times P_n$ has at most three crossings on its edges, the drawing of the resulting graph has at most one crossing on the edges of G_1^j. Contradicts to Lemma 2.2.

Next, we show that no black edge of G_1^i crosses with a red edge of $L(t-1,t)$ for $t \neq i$ and $t \neq i+1$. If not, suppose that in D there is a black edge of G_1^i, ($i \neq t$ or $i \neq t-1$) crossing with a red edge of $L(t-1,t)$. Then the red edge crosses at least two black edges of G_1^i, for otherwise, in D, the subdrawing $D(G_1^i)$ separates two G_1 and G_1^i is crossed by all six edges of $L(t-1,t)$, a contradiction. Therefore, the red edge crosses at least two black edges of G_1^i. Thus, D contains a subdrawing of a graph homeomorphic to $G_1 \times P_2$ induced by $V(G_1^{i-1}) \bigcup V(G_1^i)$ or $V(G_1^i) \bigcup V(G_1^{i+1})$ with at most one crossing on the edges of G_1^i. Also contradicts to the Lemma 2.2.

For $i = 2, 3, \cdots n-1$, let

$$Q^i = \langle V(G_1^{i-1}) \bigcup V(G_1^i) \bigcup V(G_1^{i+1}) \rangle_{G_1 \times P_n}.$$

Thus, Q^i has six red edges in each of $L(i-1,i)$ and $L(i,i+1)$, and ten black edges in each of G_1^{i-1}, G_1^i and G_1^{i+1}. Note that Q^i is homeomorphic to $G_1 \times P_3$. See Fig.2 for details.

Denote by Q_c^i the subgraph of Q^i obtained by removing nine edges u_2u_3, u_3u_4,u_4u_6,v_2v_3, v_3v_4, v_4v_6, w_2w_3, w_3w_4 and w_4w_6. Notice that Q_c^i is homeomorphic to $K_{2,3} \times S_2$, such as shown in Fig.2.

In a good drawing of $G_1 \times P_n$, define the force $f(Q_c^i)$ of Q_c^i to be the total number of crossing types following.

(1) a crossing of a red edge in $L(i-1,i) \bigcup L(i,i+1)$ with a black edge in G_1^i;

(2) a crossing of a red edge in $L(i-1,i)$ with a red edge in $L(i,i+1)$;

(3) a self-intersection in G_1^i.

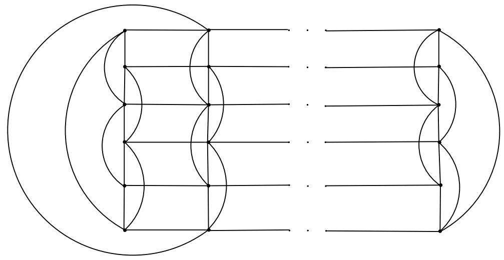
The total force of the drawing is the sum of $f(Q_c^i)$ for $i = 2, 3, \cdots n-1$. It is readily seen that a crossing contributes at most one to the total force of a drawing.

Consider now a drawing D_c^i of Q_c^i induced by D. As we have shown above, in D_c^i no two black edges of different G_1^x and G_1^y, for $x, y \in \{i-1, i, i+1\}$ cross each other, no red edge of $L(i-1,i)$ crosses a black edge of G_1^{i+1} and no red edge of $L(i,i+1)$ crosses a black edge of G_1^{i-1}. Thus, we can easily see that in any optimal drawing D_c^i of Q_c^i there are only crossing of types (i) , (ii) or (iii) above. This implies that in D, for every i, $i = 2, 3, \cdots n-1$, $f(Q_c^i) \geq cr(K_{2,3} \times S_2) = 4$ ([7]), and thus the total force of D is $\sum_{i=2}^{n-1} f(Q_c^i) \geq 4(n-2)$.

By lemma 2.2, in D there are at least two crossings on the edges of G_1^1 and at least two crossings on the edges of G_1^n. None of these crossings is counted in the total force of D. Therefore, in D there are at least $\sum_{i=2}^{n-1} f(Q_c^i) + 4 \geq 4(n-1)$ crossings. $\square$

Theorem 2.1 *$cr(G_1 \times P_n) = 4(n-1)$, for $n \geq 1$.*

Proof The drawing in Fig.3 shows that $cr(G_1 \times P_n) \leq 4(n-1)$ for $n \geq 1$.



$G_1 \times P_n$

Fig.3

We prove the reverse inequality by the induction on n. First we have $cr(G_1 \times P_1) = 4(1-1) = 0$. So the result is true for $n = 1$. Assume it is true for $n = k$, $k \geq 1$ and suppose that there is a good drawing of $G_1 \times P_{k+1}$ with fewer than $4k$ crossings. By Lemma 2.3, some G_1^i must then be crossed at least four times. By the removal of all black edges of this G_1^i, we obtain either

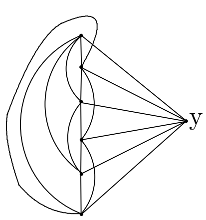
(1) a graph homeomorphic to $G_1 \times P_k$ for $i = 2, 3, \cdots n-1$; or

(2) a graph which contains the subgraph $G_1 \times P_k$ for $i = 1$ or $i = n$.

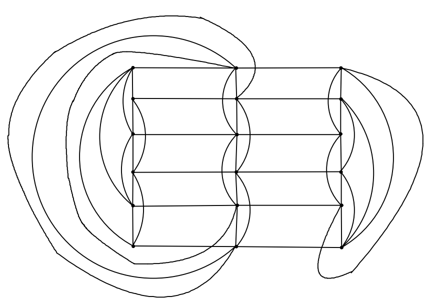
The drawing of any of these graphs has fewer than $4(k-1)$ crossings and thus contradicts the induction hypothesis. □

§3. The crossing number of $G_2 \times P_n$

By joining all 6 vertices of G_2 to a new vertex y, we obtain a new graph denoted by G_2^*.



G_2^*



$G_2 \times P_3$

Fig.4

Lemma 3.1 $cr(G_2^*) = 3$.

Proof A good drawing of G_2^* in Fig.4 shows that $cr(G_2^*) \leq 3.|V(G_2^*)| = 7,\ |E(G_2^*)| = 18.$ Apply

$|E| \leq 3|V| - 6,$

$|E(G_2^*)| + 2 \times cr(G_2^*) \leq 3 \times (|V(G_2^*| + cr(G_2^*)) - 6,$

it follows that $cr(G_2^*) \geq 3$. Therefore $cr(G_2^*) = 3$. □

Lemma 3.2 *In any good drawing of $G_2 \times P_n$, $n \geq 2$, there are at least three crossings on the edges of G_2^i for $i = 1, 2, \cdots n$.*

Proof Using the same way as in the proof of Lemma 2.2 just instead of G_1^i by G_2^i), we can get the result. □

Lemma 3.3 *If D is a good drawing of $G_2 \times P_n$ in which every copy of G_2 has at most five crossings on its edges, then D has at least $6(n-1)$ crossings.*

Proof Let D be a good drawing of $G_2 \times P_n$ in which every copy of G_2 has at most five crossings on its edges. We first show that in D no black edges of G_2^i crosses with any black edges of G_2^j for $i \neq j$. if not, suppose there is a black edge of G_2^i crossing with a black edge of G_2^j. Since D is a good drawing and there are four disjoint paths between any two vertices in G_2, there are at least four crossings on the edges of G_2^j crossed with edges of G_2^i. Now delete the black edges of G_2^i. Then the resulting graph is either

(1) homeomorphic to $G_2 \times P_{n-1}$ for $i = 2, 3, \cdots n-1$; or

(2) contains a subgraph homeomorphic to $G_2 \times P_{n-1}$ for $i = 1$ or $i = n$.

Since every copy of G_2 in $G_2 \times P_n$ has at most five crossings on its edges, the drawing of the resulting graph has at most one crossing on the edges of G_1^j. Contradicts to Lemma 3.2.

Next, we show that no black edge of G_2^i is crossed by a red edge of $L(t-1, t)$ for $t \neq i$ and $t \neq i+1$. If not, suppose that in D there is a black edge of G_2^i, ($i \neq t$ or $i \neq t-1$) crossed by a red edge of $L(t-1, t)$. Then the red edge crosses at least four black edges of G_2^i, for otherwise, in D, the subdrawing $D(G_2^i)$ separates two G_2 and G_2^i is crossed by all six edges of $L(t-1, t)$, a contradiction. Therefore, the red edge crosses at least four black edges of G_2^i. Thus, D contains a subdrawing of a graph homeomorphic to $G_2 \times P_2$ induced by $V(G_2^{i-1}) \bigcup V(G_2^i)$ or $V(G_2^i) \bigcup V(G_1^{i+1})$ with one crossing on the edges of G_2^i at most. Contradicts to Lemma 3.2.

For $i = 2, 3, \cdots n-1$, let

$$Q^i = \langle V(G_2^{i-1}) \bigcup V(G_2^i) \bigcup V(G_2^{i+1}) \rangle_{G_2 \times P_n}.$$

Thus, Q^i has six red edges in each of $L(i-1, i)$ and $L(i, i+1)$, and twelve black edges in each of G_2^{i-1}, G_2^i, and G_2^{i+1}. Note that Q^i is homeomorphic to $G_2 \times P_3$. See Fig.4 for details.

It is easy to see that $G_2 \times P_3$ contains a subgraph homeomorphic to $G_1 \times P_3$, denoted by Q_c^i. In a good drawing of $G_2 \times P_n$, define the force $f(Q_c^i)$ of Q_c^i to be the total number of crossing types following.

(1) a crossing of a red edge in $L(i-1, i) \bigcup L(i, i+1)$ with a black edge in G_2^i;

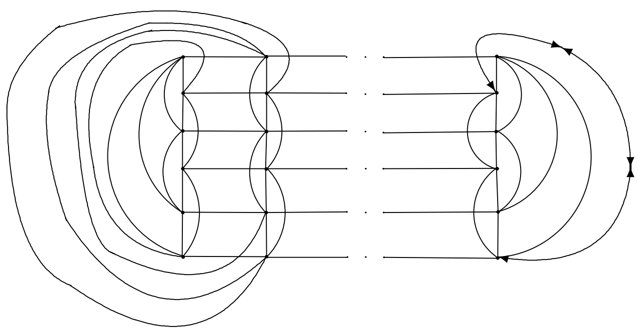
(2) a crossing of a red edge in $L(i-1, i)$ with a red edge in $L(i, i+1)$;

(3) a self-intersection in G_2^i.

The total force of the drawing is the sum of $f(Q_c^i)$ for $i = 2, 3, \cdots n-1$. It is readily seen that a crossing contributes at most one to the total force of the drawing.

Consider now a drawing D_c^i of Q_c^i induced by D. As we have shown previous, in D_c^i no two black edges of G_2^x and G_2^y, for $x, y \in \{i-1, i, i+1\}$ cross each other, no red edge of $L(i-1, i)$ crosses with a black edge of G_2^{i+1} and no red edge of $L(i, i+1)$ crosses with a black edge of G_2^{i-1}. Thus, we can easily see that in any optimal drawing D_c^i of Q_c^i there are only crossings of types (i), (ii) or (iii) above. This implies that in D, for every i, $i = 2, 3, \cdots n-1$, $f(Q_c^i) \geq cr(G_1 \times P_3) = 8$, and thus the total force of D is $\sum_{i=2}^{n-1} f(Q_c^i) \geq 8(n-2)$.

By lemma 2.2, in D there are at least three crossings on the edges of G_2^1 and at least three crossings on the edges of G_2^n. None of these crossings is counted in the total force of D. Therefore, there are at least $\sum_{i=2}^{n-1} f(Q_c^i) + 6 \geq 6(n-1)$ crossings in D. □



$G_2 \times P_n$

Fig.5

Theorem 3.1 *$cr(G_2 \times P_n) = 6(n-1)$, for $n \geq 1$.*

Proof The drawing in Fig.5 following shows that $cr(G_2 \times P_n) \leq 6(n-1)$ for $n \geq 1$. We prove the reverse inequality by the induction on n. First we have $cr(G_2 \times P_1) = 6(1-1) = 0$. So the result is true for $n = 1$. Assume it is true for $n = k$, $k \geq 1$ and suppose that there is a good drawing of $G_2 \times P_{k+1}$ with fewer than $6k$ crossings. By Lemma 2.3, some G_2^i must then be crossed at least six times. By the removal of all black edges of this G_2^i, we obtain either

(1) a graph homeomorphic to $G_2 \times P_k$ for $i = 2, 3, \cdots n-1$; or

(2) a graph which contains the subgraph $G_2 \times P_k$ for $i = 1$ or $i = n$.

The drawing of any of these graphs has fewer than $6(k-1)$ crossings and thus contradicts the induction hypothesis. □

References

[1] Bondy J. A. and Murty U.S.R., *Graph Theory with Applications*, New York: Macmilan Ltd Press, 1976.

[2] L.W.Beineke and R.D.Ringeisen, On the crossing numbers of product of cycles and graphs of order four, *J.Graph Theory*, 4 (1980), 145-155.

[3] A.M.Dean, R.B.Richter, The crossing number of $C_4 \times C_4$, *J.Graph Theory*, 19(1995), 125-129.

[4] Y.H.Peng, Y.C.Yiew, The crossing number of $P(3,1)\times P_n$, *Discrete Mathematics*, 306(2006), 1941-1946.

[5] F.Harary, P.C.Kainen, A.J.Schwenk, Toriodal graphs with arbitratily high crossing numbers, *Nanta Math.*, 6(1973), 58-67.

[6] M.Klesc, On the crossing numbers of Cartesian products of paths and stars or cycles, *Math. Slovaca*, 41(1991), 113-120.

[7] M.Klesc, On the crossing numbers $K_{2,3} \times S_n$, *Tatra Mountains Math Publ.*, 9(1996), 51-56.

[8] M.Klesc, The crossing numbers of $K_5 \times P_n$, *Tatra Mountains.Publ.*, 18(1999), 63-68.

[9] R.D.Ringeisen, L.W.Beineke, The crossing number of $C_3 \times C_n$, *J.Combin.Theory*, Ser.B, 24(1978), 134-136.

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Research papers

[8]K. K. Azad and Gunjan Agrawal, On the projective cover of an orbit space, *J. Austral. Math. Soc.* 46 (1989), 308-312.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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